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**SHIFTING SANDS: STUDENTS'
UNDERSTANDING OF THE ROLES OF
VARIABLES IN 'A' LEVEL MATHEMATICS**

**Thesis submitted for the degree of
Doctor of Philosophy
in
Mathematics Education**

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Abstract

The move from GCSE to 'A' level mathematics in schools in England and Wales can be, at least partly, characterised by an increase in abstraction and sophistication associated with the introduction of more variables. I have described as 'second variable' situations those in which letters were used in roles that went beyond the single unknown value or the dependent and independent variable, for example

- the equation $y = mx + c$ as a general equation of a straight line
- the use of (a, b) to describe a general point.

This thesis attempts to describe the experiences of students in meeting these situations.

The data which I present are drawn from a variety of sources, but come chiefly from a year spent teaching and observing two 'A' level mathematics classes in two different schools. Other sources are my own mathematical work and that of colleagues, in particular two groups of teachers with whom I met to discuss my research. I also refer to my teaching of other groups of students.

My conclusions

- distinguish between structural and empirical generalisation
- identify the shifts in the roles of literal symbols which take place in the solving of some types of problem
- describe how stereotyping affects students' treatment of literal symbols and assists in or interferes with solving of problems
- list some components of 'second variable thinking'.

My research method is qualitative rather than quantitative. It draws on my teaching experience and makes a virtue of the subjectivity of both the researcher and the reader. I offer a number of mathematical exercises to the reader and intend that he should draw on his experience of these exercises in interpreting the thesis. I expect the validity of the thesis to be judged by its coherence and by its capacity to inform the future practice of myself and of readers.

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My mother and husband, as final proof readers, deserve my thanks for their diligence and patience. Finally, I express my thanks to my family who have tolerated four years of systematic neglect without complaint in order to allow me the space to complete this work.

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Introduction

The school curriculum for England and Wales has a natural break at the end of compulsory schooling at age sixteen. After taking GCSE examinations at this age students can go into employment, take vocational training at a greater or lesser remove from the workplace, or may choose the academic route by taking Advanced Level ('A' level) courses. Those who follow the academic route opt for three or four subjects of which one or two may be mathematical. For this reason the beginning of the sixth form course marks a break from what came before. Students are in new classes, often in new institutions and with new teachers. In their mathematics lessons they are grouped only with those who have chosen to study mathematics at a more advanced level but who may have very different background experiences in mathematics.

The move from compulsory mathematics to 'A' level mathematics is often seen by both teachers and students as a difficult one to make. Indeed a large number of students fail to adjust to the changes and drop out of courses at an early stage.

Whilst the social and institutional changes which occur at this stage might be used to account in part for the difficulties experienced by students, it is also worth examining the changes in subject matter and approach which characterise the move to 'A' level.

This move is marked by an increase in the degree of abstraction and of sophistication. Three examples will serve to indicate what I mean by these words.

First, at the higher levels of GCSE students meet the equation $y = mx + c$ as a descriptive form. They would know that, in this form, m stands for the gradient and c for the intercept with the y axis. They would know that, for example, the line $y = 3x + 1$ has gradient 3 and intercept 1. At 'A' level, students would use the form $y = mx + c$ in order to derive the equation of a straight line fulfilling certain conditions. For this they must substitute into the form and manipulate the resulting equation. This is an increase in the level of sophistication. Later they must substitute literal symbols (that is letters which stand for quantities) rather than numbers into the form. This constitutes an increase in the level of abstraction.

Secondly, at GCSE students solve problems by forming and solving linear or quadratic equations. In this way they are introduced to the idea of naming the unknown, expressing in an equation the conditions which constrain it, and solving this equation. In other words, they learn the 'analytic method' (Polya 1945). At 'A' level this idea is extended to naming an unknown point and expressing the conditions which constrain it in order to form the equation of a locus. The aim now is to find a relationship between two variables rather than finding a numerical value for a single variable. The object of their search then is an equation representing a curve, and is more abstract than a value for a variable.

Thirdly, at GCSE students are required to solve quadratic equations by factorising, by completing the square or by applying the quadratic formula. They may see the formula derived but would not be expected to derive it for themselves. At 'A' level students would be expected to achieve a comprehensive understanding of the nature of the roots of quadratics. This involves seeing the connection between the number of points of intersection of the graph with the x axis, and the number of real roots of the equation. It involves seeing how the number of real roots is indicated by the value of the discriminant. It involves seeing how the values of the roots are related to the coefficients of the quadratic.

I see in this example two significant moves towards greater abstraction and sophistication. The first is from using a formula by substituting values into it, to manipulating and interpreting the formula. At GCSE students substitute numerical values into the formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve particular equations.

At 'A' level they look at part of the formula, $(b^2 - 4ac)$, and what it indicates. They may substitute literal symbols into this expression in order to draw conclusions about the values represented by those literal symbols under certain conditions. This is similar to the move in the use of $y = mx + c$. The second

move is from solving particular equations to seeing them as representative of types of equation. The 'A' level student needs to be able to understand why factorisation fails as a method for solving equations with no real roots, rather than just seeing that the quadratic cannot be factorised.

Each of these developments from GCSE level to 'A' level involves the introduction of more variables, often used in roles which were previously taken by numerical values. This is a crucial aspect of the change of level. For this reason a study of students' early experiences of problems which involve more than one variable can play an important part in helping us to understand the transition from GCSE to 'A' level. It was my experience of sixth form students' struggle to cope with this transition which led me to begin this research.

The thesis begins, in chapter one, 'Beginnings', with a description of the specific events which led me to adopt this area as an object of study.

It goes on to look at other work relevant to my interest. There is remarkably little existing research which addresses this area directly and therefore few research-based models which can be used by teachers in forming an understanding of their students' actions. A great deal has been written more generally, however, on the subject of students' learning of algebra. Some of the research and other writing which I found useful in framing the way I thought about these issues is described in chapter two, 'Sources of Influence'.

My experience as a classroom teacher has given me a particular perspective on the relevance of research to the act of teaching. My concern to remain true to my identity as a teacher throughout the research process and to produce a thesis which would be perceived as relevant and useful by other teachers has heavily influenced my research methods. In chapter three, 'Methodology', I set out the reasoning which led me to adopt the methods I did, and in chapter four, 'Description of Method' I outline the steps I undertook.

In chapters five to eight I lay out the major themes, awarenesses and models which have arisen from my study, alongside accounts of incidents which raised them to my mind and descriptions of tasks which the reader is invited to undertake in order to experience the points I am making.

Chapter five, 'Particular and General', looks at ways in which students arrive at generalisations and the part played by teachers in these processes. It also addresses the nature of the relationship between particular and general in mathematics, especially in cases where the general is expressed in terms of more than one variable.

Chapter six, 'Roles of Literal Symbols', explores the use of symbols in roles which are neither particular nor general. It gives some frameworks for understanding the ways in which ambiguity in relationships between variables causes confusion for students.

In chapter seven, 'Patterns of Problem-Solving', I describe some ways in which the students' struggle to deal with the complexity of relationships becomes apparent during problem-solving, and diverts their attention from the original task.

Ways in which an awareness of these issues might inform classroom practice are explored in the first section of chapter eight, 'Analysis of Two Problems', where I apply some of the metaphors developed in chapters five to seven to a particular lesson. In the second section of the chapter I look in detail at a mathematical task which was particularly influential in this study. The work of teachers and several students on this task is analysed in the light of the earlier chapters.

Finally chapter nine, 'Summary and Conclusions', draws together the themes of chapters five to eight and looks back to the questions which were left pending in chapters two and three.

Chapter 1 Beginnings

On one occasion in the autumn of 1992, when I was employed as a maths teacher in a comprehensive school, my sixth form class was working on the following problem:

Problem A For which values of k is $k(k - 1)x^2 + 2(k + 3)x + 2$ positive for all real values of x ?

Many declared that they did not know how to answer it. A suggestion to use the discriminant flitted around the class and the formula $b^2 - 4ac$ was duly recalled and applied. But what to do with it?

'A range of values is involved. Something must be always positive. Should we put the discriminant greater than or equal to zero? We'll try that.'

'Our answer is exactly the opposite of what is given in the book. Have we made a mistake and forgotten to change the sign somewhere?'

As a teacher I had been aware beforehand that this question was likely to prove difficult, but I was still perplexed by the students' responses to it. I wanted to help them see what I saw in the question.

When I looked at the expression $k(k - 1)x^2 + 2(k + 3)x + 2$ I saw a family of functions, each value of k representing one function in the family. I also saw a family of curves, each representing one of the members of the family of functions. I saw the question as asking me which of the family members was represented by a curve that stayed always above the x -axis.

In the summer of 1993 I was no longer a full-time teacher but I had the opportunity to spend an afternoon working with two sixth form classes normally taught by a colleague. I chose to ask them to work on the same problem.

The students had available some graphic calculators and I decided to encourage them to plot graphs of the expression for various values of k because I hoped that this would help them to see what I saw.

Two things surprised me in what I saw these students doing. First, a number of them eventually succeeded in the task without having used the graphic calculators and without having seen their relevance to the question. Secondly, four students, working in two pairs, took a very different approach. They selected values for k and for x and evaluated the expression to see if it was positive or negative in each case.

I was trying to encourage the students to see through the general expression $k(k-1)x^2 + 2(k+3)x + 2$ given in the question to the particular functions which it summarised. I hoped that this would enable them to see which functions satisfied the condition that they be always positive and to move back to the general by stating which values of k corresponded to these. Some students solved the problem in general without looking at particular functions while others started at a level of even greater specialisation, that is they began to look not at particular functions but at particular values of particular functions. These students did not manage, in that lesson, to increase the level of generality through which they viewed the situation to one which allowed them to see a method of solution.

My experience with this class set me on two lines of investigation which I outline below. One was of problems involving *second variables*. I identified that the students had merged two inequalities which were associated with this problem i.e. $k(k-1)x^2 + 2(k+3)x + 2 > 0$ and $b^2 - 4ac < 0$, to form $b^2 - 4ac > 0$ (into which they substituted expressions involving k) and thought of this as the *double-layered* nature of the problem. I understood this *double-layeredness* to be a result of the presence of x and k in this problem and wanted to consider further the nature of the confusion caused by this kind of situation. The second line of investigation was of movement between particular and general. I noted the differences between my suggested solution procedure (using the graphic calculator) and the routes chosen by the students in terms of the use they made of particular and general and I began to seek out further situations through which I could study this issue.

My conception of problems involving *second variables* included many situations which are found in the first year of sixth form courses. By the end of a GCSE course students are reasonably familiar with the use of a variable as an unknown in equations-to-solve and as dependent and independent variables in function rules. I recognised as *second variable* situations anywhere where letters were used in roles that went beyond the single unknown value or the dependent and independent variable. For example:

- the equation $y = mx + c$ as a general equation of a straight line
- the 'standard' quadratic expression $ax^2 + bx + c$
- the formula for the coordinates of a point dividing a line, given by the coordinates of its endpoints, in a given ratio.
- equations for a curve in 'parametric' form e.g. $x = t^2, y = 2t$
- vector equations of curves of the form $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$
- representatives of sets of elements e.g. the set of matrices of the form

$$\begin{pmatrix} a & a^2 \\ 1-a & a \end{pmatrix}$$

All of these situations could be described by the term *family* - a family of curves, of points, of elements. In most cases the second variable could be referred to as a parameter. Nevertheless I shall not use the word 'parameter' to describe this set of situations. It has a number of technical definitions (see appendix A) and will not convey the meaning I intend in most cases. In chapter 2 I discuss terminology and the use I will make of the words 'unknown', 'parameter' and 'variable'.

I recognised that there were many such situations in the first year of an 'A' level mathematics course. I knew that my 'expert' view of these situations was very different from the way in which the students saw them and I wanted to know what were the students' experiences in facing such questions. I also wanted to know how teachers could come to appreciate their own mathematical awarenesses concerning such situations and to be sensitive to the students' experiences.

The second area I termed movement between particular and general. I interpreted the behaviour of the students who were examining the values of the expression for particular values of x and k as having been trained by their experience of GCSE 'investigations'. Both pairs who adopted this strategy were trying to spot patterns in tables of results and proved quite resistant to advice to change their approach. There was also concern expressed in the

mathematics education press at that time about the effects of 'investigational work' on students beginning 'A' level. However these mostly concentrated on the perceived deficit in students' attainment because 'investigations' had taken the place of more traditional drill and practice algebra. I was more interested in the potential of these tasks to develop an awareness of particular and general than in their effect on students' fluency in algebraic manipulation. I began to look therefore at problems which might be offered as investigative tasks. Typically these problems would offer a general question which could be approached by looking first at particular cases. For example

Problem B Suppose that n (≥ 4) points are chosen on a circle, and each pair of points is joined by a straight line. Assume that no three lines meet at a point except on the circle. Let p_n be the total number of triangles formed within the circle. Find a formula for p_n for $n > 6$

(Adapted from 'The Colleges of Oxford University, Entrance Examination in Mathematics', 1989 Paper II question 14)

I was interested in the awarenesses of the relationship between particular and general that they offered. I chose tasks that were of interest to me and would be of interest to teachers at their own mathematical level, rather than at the students' level. My aim was to look at my own attention on particular and general in solving these problems and to offer my colleagues opportunities to do the same. Further examples of these can be found in later chapters.

As I pursued these lines of investigation I found that there was a uniting theme. The theme was shift of attention between particular and general. My interests included shifts in both directions. I was concerned with, for example, seeing $y = mx + c$ as a particular straight line (seeing the particular in the general) and with finding a formula for p_n (in the example given above) by studying the case $n = 7$ (seeing the general in the particular).

The questions then with which I began my study were:

What are students' experiences of facing second variable situations?

What mathematical awarenesses are available through these situations?

What techniques are available to teachers for working with students on these awarenesses?

What is involved in shifting attention from particular to general and from general to particular?

The way in which I intended to pursue answers to these questions, as well as the questions themselves, was influenced by my experience and self-image as a mathematician and as a teacher.

I wanted to maintain both of these roles in the work I did. It was important that I build on my awarenesses as a teacher with ten years' experience and on my awarenesses as a mathematician. I chose therefore to adopt the role of teacher in my work with students. My intention was to focus on the mathematical awarenesses they exhibited and on what the teacher could do to help students work on them.

Chapter 2 Sources of Influence

In chapter one I described what led me to follow up two lines of enquiry, that of second variables and that of movement between particular and general. In this chapter I outline the research reports and other writings in these and related areas which have influenced the formation of my theoretical frameworks and research questions.

I include writings which could not strictly be described as research reports whenever they have significantly influenced my thinking and, in my treatment of them within this chapter, I make no distinction between different types and purposes of writing.

Many of the writings to which I refer have influenced me because I made connections between them and my classroom or mathematical experiences. These connections are for the most part elaborated in the chapter which describes the experience, rather than in this chapter.

Research on Second Variables

The Notion of Variable

It seems appropriate to begin with a discussion of vocabulary. Three words very frequently used in this context by teachers, pupils and researchers are 'variable', 'parameter' and 'unknown'. Other writers have found it necessary to distinguish more clearly than does common usage the meanings of these terms. For example, Harper (1987) distinguishes between 'unknown' and 'given' as between *potential determination* and *assumed range of values*.

A number of writers have explored the complexity of language connected with notions of variable and of the use of letters (literal symbols) to stand for

quantities. Philipp (1992) lists seven uses of literal symbols, amongst them unknowns (e.g. x in $5x - 9 = 91$), varying quantities (e.g. x, y in $y = 9x - 2$) and parameters (e.g. m, b in $y = mx + b$).

Schoenfeld and Arcavi (1988) also consider the multiple meanings of the word 'variable' but without attempting a catalogue. They content themselves with describing some exercises designed to draw the attention of teachers to multiple meanings and with suggesting approaches which might help students to come to terms with the complexity. In doing this they present the reader with some of the experiences which they themselves have found helpful in expanding their awarenesses of the notion of variable, rather than trying to convey those awarenesses through exposition in what they write.

Radford (1996) discusses the difference between unknown and variable in a historical context through the work of Diophantus. For Radford the key feature of the concept of variable is that it is a dynamic quantity 'which can have different values depending on the values taken by another quantity' (p50). It is used to establish relationships between numbers rather than to solve word problems. He suggests that a historical perspective might be used in introducing an appropriate distinction between unknown and variable in the teaching of algebra.

Frege (see Geach 1980) dismisses the idea of a variable by concluding that 'The word variable has no justification in pure Analysis' (p111). His principal objections are (i) that variation must be in time or, if not, in what? and (ii) that a variable cannot be a variable number because a number which changes is no longer the same number. His concerns might not shed much light on the process of learning the meanings of variable, but they do give further evidence of the complexity of that process and hint at some of the conceptual obstacles which students might encounter.

It is tempting to give descriptions, if not definitions, of the way in which I intend to use these words. However I do not believe it would be helpful for me to give such descriptions for the following reasons.

First, to make a firm distinction would imply that there are definitions generally agreed by the mathematics and mathematics education communities. I do not believe this to be so, and find rather that the words have to be understood in the context of a particular mathematical task or situation.

Secondly, these three words, but perhaps most particularly the word 'parameter', hold meanings for sixth form students which are connected with only one or two particular contexts. In the case of 'parameter' this is the context of equations of curves expressed in parametric form. In learning about such

equations they necessarily come across the word 'parameter' (for example in answering the question 'Show that the curve with parametric equations $x = 1 + 4\cos\theta$, $y = 2 + 3\sin\theta$ is an ellipse'), whereas in other situations in which 'parameter' might appear its use is not essential and it is frequently omitted. It is rare to find the word in an 'A' level text book in any context other than parametric equations for a curve. The word 'unknowns', on the other hand, usually summons up systems of linear equations and a description of the kind 'three equations in three unknowns'. The word 'variable' is often associated with situations where calculus is applied to contexts beyond y as a function of x . The use of the word is common when there is a need to distinguish between variable and constant in a function which is to be differentiated.

Thirdly, these three words cannot cover all shades of meaning in different mathematical situations. For instance, in the question 'Find the value of k such that $3x^2 + kx + 12 = 0$ has equal roots', I might first interpret k as a parameter whose variation produces a family of quadratic equations. Later on I need to see k as an unknown whose numerical value I must find. The role of k could be described both by 'unknown' and by 'parameter', but not adequately by either alone. A further interpretation of the role of k in this question is that its role changes from that of parameter to that of unknown as the solution emerges. Bloedy-Vinner (1994) gives another example of an interpretation of a problem which has a literal symbol changing its role in the course of the solution:

'Solving this problem ("Find an equation for the line through (2, 5) with slope 3") starts with writing an equation $y = ax + b$, where common knowledge determines that x and y are variables whereas a and b are parameters. The process continues by substituting the constant 3 for a , and solving an equation with unknown b , where constants are substituted for x and y . The process terminates by substituting the constants found for a and b , and by letting x and y be variables in $y = 3x + 1$ ' (p89-90)

Here the role of b changes from that of 'parameter' to that of 'unknown' as the solution progresses.

I intend, then, to use the words 'parameter' and 'unknown' only as having meaning within particular mathematical situations. Within these situations I shall describe briefly what I intend by the use of the terms.

I find, however, that I need a word to describe the category of uses of literal symbols which includes anything which might be described as 'variable', 'parameter' or 'unknown'. I will therefore use the word 'variable' in a wide variety of situations, from general to particular, to mean broadly 'letter standing

for quantity', rather than its narrower meaning as expressed by Phillip in 'varying quantity'.

Students' Difficulties with Second Variables

A small amount of work has been done specifically on students' difficulties with second variables. The secondary phase of the APU survey (APU (Assessment of Performance Unit) 1980) produced some evidence that use of letters beyond solution of equations involving one variable was relatively difficult for students.

'The facilities of items testing pupils' abilities to solve linear equations ranged from about 90% to just under 40%. The most difficult item here was the solution of a literal equation (i.e. one involving only letters)' p41

The number of such test items, though, was very small and the evidence produced was therefore limited.

In an article relevant to my theme of second variables, Adda (1982) takes as her theme synonymy and homonymy. By 'synonymy' she means the situation in which two or more symbols refer to the same object e.g. ' $\frac{4}{2}$ ' and '2'. By 'homonymy' she means the situation where one symbol can refer to more than one object. For instance, in the sentence 'To multiply a number by ten, add a zero', a confusion arises between the number as mathematical object and the symbolic representation of the number. She points out some of the potential confusions surrounding the use of letters to stand for numbers, and in particular the use of letters for different purposes within the same equation, taking as her example ' $ax^2 + bx + c = 0$ '. In this equation x represents an unknown, that is its synonym is to be found, and a , b and c represent parameters, that is they are to appear in the synonym (or expression) for the unknown. This equation could be a second degree equation in x or a first degree equation in a , b or c . I discuss students' ability to cope with these different roles within one equation in chapter six.

Schoenfeld, Smith and Arcavi (1993) give an example of a related confusion in the work of one individual student who was using a computer package and trying to reconstruct her memory of the form $y = mx + b$ for the equation a straight line. At some point in her work on this she says:

'OK so then .. um this (points to the b slot in the equation) is the y -intercept, so what's this (points to the y in the equation)?' (p151)

I suggest that the confusion here could be described as being produced by homonymy. At one point y is the symbol for a varying quantity (in $y = mx + b$),

and at another it is the symbol for an unknown (for example in $x = 0 \Rightarrow y = m \times 0 + b$)

The work described by these authors is interesting to me not only because of its reference to a student dealing with the equation $y = mx + b$ but also as a result of the similarity of the incident to one which I witnessed, and in view of the method used by the researchers. I refer to it twice more in chapter three.

Sfard and Linchevski (1994) link the ability to deal with literal symbols which take the role of parameters with a move to 'functional' thinking as opposed to 'fixed-value' thinking. Taking the question 'Is it true that $k - y = 2$, $x + y = k$ has a solution for every value of k ?' as an example, they claim that

'to understand the question, one must realise that each of the equations $k - y = 2$, $x + y = k$ represents a whole family of linear functions (or, to put it in different terms, it expresses a family of infinite sets of ordered pairs of numbers), and that for different values of k the system will yield different pairs of such values' p218

I would dispute that it is necessary to visualise the problem in this way in order to understand it. It is also possible to view the situation as a single curve defined by a pair of parametric equations (x and y defined in terms of k), which may or may not have points defined for every real value of the parameter. Equally I may solve the equations simultaneously in terms of k and find that it is possible to do so for every value of k , without having any graphical image. Again, I might see the equations only as a pair of equations in three unknowns. However I would agree that, in order to solve this problem, it is necessary to acknowledge the different roles of k and x and y in it, even though this acknowledgement may be only implicit in the treatment and not explicitly stated.

In my analysis of students' work I have avoided stating what kind of thinking is required in order to solve a particular problem. Rather I have aimed to describe what kind of awarenesses are available in a certain mathematical situation and what kind of awarenesses the student exhibits.

Furinghetti and Paolo (1994) and Bloedy-Vinner (1994) both undertook larger scale studies which involved setting questionnaires or tests to 82 and 199 students respectively. Each focuses explicitly on 'parameter', which is described by Furinghetti (p368) as 'an elusive concept that carries with it the difficulties of literal symbols and the ambiguities of its analogy-difference with the concepts of variable and unknown'. Bloedy-Vinner conveys her usage of the term by a series of examples of questions involving parameters which might be encountered by high school students. Each author hoped to test her own

formulation of the skills and thought processes involved in solving such problems by looking at the responses of the students to the test questions.

Bloody-Vinner (1994) describes a study which was part of a larger project concerning the nature of algebraic language. She describes 'failure to understand algebraic language' as 'analgebraic thinking'. In exploring the difficulties students have with 'parameters' (the meaning of this term is conveyed by her list of examples) she uses the notion of a quantifier structure. For example, in the question, 'For which values of m does the equation $m(x - 5) = m + 2x$ have no solutions?', the quantifier structure is 'There exists m such that there does not exist x such that the equation holds'. That is, m is quantified before x . In the analysis of students' responses to test items, written answers are categorised by the researcher's interpretation as algebraic or analgebraic. This distinction is equated with the distinction between correct and incorrect ordering of the quantifiers. Her conclusion is that there is a very high incidence of analgebraic thinking.

I would take issue with some of the underlying assumptions of this research. The introduction contains the sentence 'many students do not understand algebraic language correctly' (p88), which presents all kinds of difficulties concerning what it means to 'understand correctly', even if we are content to interpret 'many' and 'algebraic language'. In addition a recurring phrase is 'the analgebraic mode of thinking' (first used p88). Again we have 'analgebraic students' and 'the purpose of this study is to examine to what extent students are algebraic or analgebraic in the context of parameter' (both p91). The implication of these phrases taken together is that an individual student is an algebraic or analgebraic thinker regardless of context, and without possibility of development of their thinking skills. No mention is made in the paper of the reliability of the test questions, that is the extent to which students were consistently algebraic or analgebraic thinkers as defined by the researcher's analysis.

I have nevertheless found it useful to draw on the idea of a quantifier structure in considering my students' work. This framework gives one means of distinguishing between roles of variables and I expand on it in chapter six. However I would not claim that understanding the quantifier structure of a problem is equivalent to understanding the algebraic language in use. Other aspects, for example freedom and constraint, or standard forms, seem to be equally important.

Furinghetti and Paolo (1994) report a similar study, in which an eleven-part questionnaire was given to 199 sixteen and seventeen year olds. Amongst the

points which arise from the researchers' analysis of the replies are the following:

- students feel confident with questions requiring computational processes
- letters in apparently symmetrical roles, e.g. $kx > 0$, cause difficulties for students
- some letters elicit a stereotyped expectation of role

Each of these points will also arise in my analysis of my students' work (in chapters seven, six and six respectively). Furinghetti and Paolo's highlighting of these points served to sensitise me to see them in my classroom and in my own mathematical work.

There is much evidence, then, to support the claim that the introduction of second variables represents an increase in the level of difficulty. However, attempts to understand the nature of students' experiences of second variable situations are divers and incomplete.

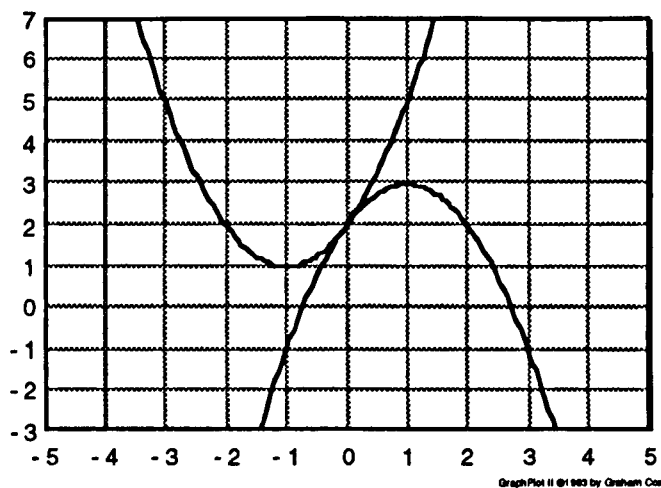
Second Variables and Graphs of Functions

A small but rapidly expanding body of work exists concerning the role of second variables in the connection between functions and their graphs (for example, the effect of varying a , b and c on the graph of $y = ax^2 + bx + c$). This work has been prompted by the increasing use in schools of graph plotters as tools to aid students in their thinking about these connections.

- One group of writings describe research projects in which experimental approaches have been used to teach students about the graphs of functions using a computer.

Artigue and Dagher (1993) describe how a computer game was used to help students learn the graphical interpretation of some second variables in various forms for a quadratic function (for example the roles of a , p and q in $y = a(x - p)^2 + q$). They found that, although the game was successful overall, knowledge gained in the computer environment was not always transferable to a more formal situation.

Of particular interest to me was an anecdote related by the researchers at the conference but not included in the paper. The story was about a computer technician who worked on the project and therefore had a lot of exposure to the game but without receiving any instruction and without a formal background in mathematics. He was quoted as having recognised from a diagram similar to that below that, in the form $y = ax^2 + bx + c$, the two curves had equal values of b and c and values of a which were equal in magnitude and opposite in sign.



Asked how he knew, he had no explanation to give. The suggestion is that the technician had achieved fluency through practice and immersion in connecting this particular formation of curves and the interpretation of the values of a , b and c placed upon it. It is not clear whether, in making the connection, the technician followed the mathematical reasoning which the researchers might have expected him to use.

This incident raises questions about the purpose and outcome of the game in terms of fluency and mathematical reasoning. Is it the purpose of the game to produce in students fluency in recognising connections between features of graphs and the values of the coefficients in their equations? Or is it to develop in students the ability to reason about these connections? Does the game provide opportunities to develop fluency without reasoning or reasoning without fluency?

Confrey (1994) describes how her work with students on transformations of graphs has led her to formulate six different teaching and learning approaches. All six approaches were developed in the context of computer graphing environments and dealt with transformations of the form $y = Af(Bx + C) + D$. The first approach is described as 'substitution to a template' and involves 'learning to identify the actions associated with each parameter' (p218). She comments that this approach is perhaps the most concise but was problematic for all but the most mathematically successful students. She attributes this difficulty in part to the lack of symmetry between actions on x and on $f(x)$.

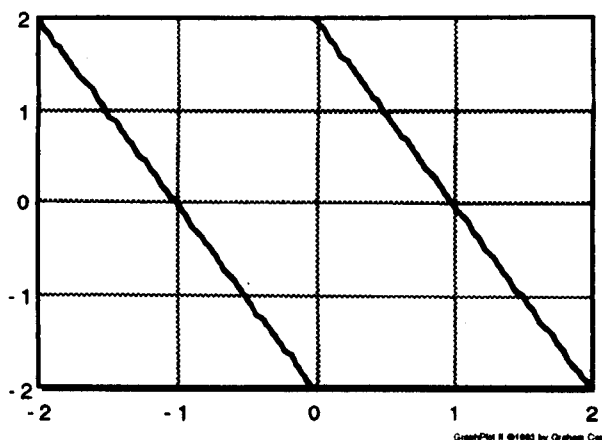
The other five approaches are less conventional. They do not make such explicit use of the form $y = Af(Bx + C) + D$ and the parameters A , B , C and D but rely on generic examples of the effects of transformations. Some rely heavily on use of the computer environment. Confrey stresses the need for these environments to be multi-representational, that is to have dynamic links between equations, tables of values and graphs of the functions. Working in

these environments she has been able to describe and refine her own thinking on transformations in the light of approaches taken by her students.

Ruthven (1990) found that students who had been taught using a graphic calculator were better at constructing the equation of a given graph than their peers who had been taught more conventionally.

Each of these studies moves us away from the simplistic notion that the use of graph plotters will enable students to learn 'better' about graphs of functions. They point out ways in which experience of graph plotters changes the way in which students learn about and hold concepts of the nature of function graphs.

Another set of writings focuses on what students and teachers see when they look at a graphing screen. Goldenberg (1988) points out some of the ways in which the viewing window used on a graph plotter can affect the student's perception of the graph. In particular he describes how choice of window can affect the way in which the transformation of one graph into another is seen. He takes as an example graphs of $y = -2x - 2$ and $y = -2x + 2$. Whereas the conventional analysis has one as a vertical translation of the other through four units, viewed through an appropriate window (see below) the appearance is of a horizontal translation.



This particular form of the phenomenon has led to suggestions that translations of graphs are best discussed in the context of quadratics or other curves initially.

In Goldenberg (1991) he speaks further about students viewing graphs on graph plotters and examines in some detail what they might see. He distinguishes between seeing the graph as a whole and seeing specific features of a graph e.g. the y -coordinate of the vertex, along the same lines as the distinctions made by Van Hiele in geometry (Wirszup 1976). The following classroom incident breathed life into this distinction for me.

In a sixth form class the students worked on the graphical interpretation of a , b and c in the equation of a parabola, $y = ax^2 + bx + c$. I showed them how to use a graph plotter and set them to investigate the meanings of a , b and c . One pair of students told me that 'when you change b it moves sort of across and up'. I asked them what was the 'it' that moved and after a moment one of them said 'the turning point'. The identification of the 'it' which moved seemed to be significant in their confidence to talk about what they had seen (they no longer claimed that it 'sort of' moved) and in their progression to connecting the movement with algebraic expressions for the coordinates of the turning point.

Magidson (1989) describes another aspect of the way in which novices see graphs differently from experts. In this study students were asked to plot $y = 2x + 1$, $3x + 1$, $4x + 1$ etc. and to comment on what they saw. The students focused on where the lines entered the computer screen and how jagged they were, rather than that they all went through the point $(0, 1)$.

These considerations of what students see when they look at a graph drawn electronically can be summed up in the difference between 'looking at' and 'looking through' (see Mason 1993). 'Looking at' implies that what is seen is what is there, and, in consequence, that every viewer sees the same. 'Looking through' points out that the image on the screen can be a vehicle for the viewer to see some generality evoked by the particular set of pixels. When we speak of a 'viewing window' for a graph we are using a metaphor which has the graph as an object existing independently of the means of representation. For students to look through the screen and see the graph as object requires them to make an act of generalisation.

My interest in graph plotters here is that they allow teachers and students to exemplify the variable nature of the second variables in forms such as $y = ax^2 + bx + c$ with relatively little effort. Plotting a large number of graphs with different values of a , b and c allows a study of the effects of varying these values. In so doing, however, we can provide a metaphor for the different roles of x and y as compared with those of a , b and c . It quickly becomes apparent that each new value of a , b or c gives rise to a new graph, whilst each new value of x or y gives rise to a new point. Choices about values of a , b and c give rise to a new equation whereas choices of values of x and y then allow the graph to be plotted. I, as user, choose to vary the values of the second variables, whereas the computer varies x and y . In the first case I am the agent of choice and in the second the computer fulfils this role. This is expressed by Menghini (1994) as follows

'the parameter is generally given on the computer keyboard and is varied manually by the user, while the variable changes according to a "cycle" carried out by the machine'. (p11)

The question of the agent of the choice of values for each type of variable, in the context of the graph plotter, offers a tangible way of distinguishing between the roles of the two types.

Such an analysis of the role of the graph plotter in developing students' understanding of second variables assumes that the varying of x and y 'by the computer' is an unproblematic issue for the students. Kieran et al (1993) report on a study involving students younger than those who would typically be taught about transformations of graphs. The study engaged students in adopting a functional approach to problem-solving using a computer package which facilitated plotting a graph. She claims that it was easier for students to see graphs as being made up of an infinite number of points if they plotted each point individually (on the computer) rather than watching the graph being drawn all at once. She draws on this finding to distinguish between understanding a graph as a process and understanding it as an object. This is a theme to which I shall return later in this chapter.

Students with whom I have worked have had increasingly easy access to graphic calculators, although limited access to graphing packages on a computer. I have been concerned to see whether the influence of their familiarity with graphic calculators is evident in their treatment of parameters used to define families of curves. This theme emerges briefly in chapter six.

Movement between Particular and General

Classical theories of the psychology of learning take as one of their foundations the human ability to distinguish, to identify sameness and difference, like and unlike, and thereby to group, separate and classify. The notion of classification allows us to conceive of members of a class and hence of representatives of classes, or examples.

The issue of classification is also a route into consideration of particular and general. 'Particular' describes features of an individual member of a class, whilst 'general' describes features common to all members. A statement describing an attribute of a particular member of a class might be adapted to describe an analogous 'general' feature of every member of the class.

Many writers, as I describe below, have considered the issue of particular and general in the learning of mathematics. In particular, several major theorists

have queried classical ideas about the nature of the relationship between general and particular in the learning process.

Classical learning theory has us forming concepts by abstraction of the commonalities from numerous encounters with the particular. Dienes (1960) bases his principles for the teaching of mathematics on the notion of abstraction from examples. Skemp (1971) uses the example of a child developing the schema of 'chair' through numerous encounters with examples and non-examples of chairs. (Bruner (1966), however, could find no evidence that non-examples were useful in concept formation).

Borasi (1984) expresses disquiet at this interpretation of learning new concepts in the context of mathematics education. She points to psychological evidence produced by Tall and Vinner (1981) which contradicts the notion that 'irrelevant' attributes of the examples from which a concept has been abstracted will be forgotten once the concept is established. They found, on the contrary, that some features of the examples which were presented in the teaching of a concept and were not attributes of the concept were nevertheless retained as part of the students' 'concept image', that is the students' mental picture of the meaning of the concept. Borasi goes on to expound the shortcomings of the abstraction model in the case of the concept of an infinite set.

Freudenthal (1978) argues that learning of mathematical concepts at school does not take place by the process of abstraction:

'the origin of general ideas, concepts, judgements and attitudes in the learning process, whether they are attained in a continuous process, by *comprehension*, that is by generalising from numerous examples, as is the common opinion, or by *apprehension*, that is by grasping directly the general situation, which is my thesis.' (p170)

He describes methods of promoting 'apprehension' in the classroom by the use of 'paradigms', that is single examples which give access to the general situation. He quotes Davydov's work (see below) as expressing some similar ideas.

Lakoff (1987) argues a similar case concerning the classical view of categorisation. He opposes to it a new theory of categorisation called 'prototype theory' in which it is claimed that

'...prototype theory....suggests that human categorisation is essentially a matter of both human experience and imagination - of perception, motor activity, and culture on the one hand, and of metaphor, metonymy, and mental imagery on the other' (p8)

Further, prototype theory suggests that there are good and bad examples of members of a category. This contradicts the classical view that no example of a category is any better than any other example and, says the author, fits better with our experience.

Rosch (1975) established that subjects could reliably rate how 'typical' an item was of a particular category under a variety of different artificial settings and in Rosch (1976) found that 'typical' items were more quickly recognised as belonging to a certain category. She also claims that on hearing category names subjects generate a representation of typical category members. The extent to which 'prototypes', or typical category members, shape students' thinking in mathematics is explored in Confrey (1991).

Davydov (1990) presents a comprehensive criticism of classical ideas on concept formation. He concludes that the traditional psychological theory of concept formation holds that it is essentially empirical. Traditional theory describes a process of inductive generalisation. Davydov rejects this as inadequate to account for all advances in knowledge. For example, the kind of advances described by Kuhn (1962) as 'scientific revolutions' fall outside its scope.

Davydov describes how traditional ideas about concept formation through encounter with examples pervade the school curriculum, taking as instances the areas of grammar and mathematics. In mathematics, too much emphasis is placed on solving particular numerical problems and not on relations between numbers. Krutetskii (1976) found that mathematically successful students were those who could see the general structure of a particular problem 'on-the-spot' without considering further problems with the same structure. Davydov's hypothesis is that by basing teaching methods on the idea that students will move from the concrete to the abstract, we create difficulties for students in making abstractions.

He describes a teaching experiment in which children at primary school are introduced to relationships between numbers without referring to particular numbers. This experiment is described more fully in Davydov (1962). Children were introduced to the meaning of ' $A = B$ ', ' $A < B$ ' and ' $A > B$ ' as a record of the result of comparing objects by weight, length etc. Later ' $A < B$ ' was replaced by ' $A + e = B$ ' and seen to be equivalent to ' $A = B - e$ ' and ' $e = B - A$ '. The children's test scores suggest a high level of mastery of this experimental syllabus.

In a report of the work of Davydov, Freudenthal (1974) suggests that the experiment lost its way after the first year because it began to concentrate too narrowly on teaching children to solve the type of word problems which were

then common on the Russian school mathematics syllabus and experienced as difficult by most children. The suggestion is that curriculum reform needs to include reform of the aims and objectives as well as teaching methods employed.

Kravin (1990) contains a report on a programme carried out in second grade in a California elementary school modelled on the suggestions of Davydov. The programme covered comparison only and the $<$, $>$ notation. No 'results' are discussed.

Kravin suggests that this method loops continually from concrete to abstract and forms a bridge between constructivist and behaviourist teaching and between exploration- and drill-based teaching.

The works of these three writers (Davydov, Krutetskii and Freudenthal) combine to suggest an alternative model for a student's journey to a generalisation. This model has students obtaining a direct grasp of the general rather than working through many particulars to achieve such an understanding. In more practically-based writing, Hewitt (1992) and Kieran et al (1993), by reference to classroom experience, find fault with classroom practice associated with an 'abstraction from particulars' model of the journey.

Hewitt describes how classroom activities set up to encourage generalisation can deteriorate into a well-worn procedure of producing a table of values and looking for a relationship between the numbers, rather than seeing any relationships in the situation which gave rise to the table of results. In other words the encouragement to use an inductive generalisation (by drawing up a table and 'looking for a pattern') can reduce the possibility of an 'on-the-spot' generalisation being made by reference to the mathematical structure of the problem.

Kieran similarly found that placing attention on a table of values reduced awareness of the original problem from which the table had been derived. In this case the table was produced by using a spreadsheet.

I will discuss these observations again in the light of my own experience in chapter five.

Other writers have expanded on traditional understandings of abstraction without stressing their limitations. Dreyfus, for example (1991) speaks of abstraction as focusing on relationships between objects rather than on the objects themselves. This description includes the traditional idea of shifting attention to the similarities and differences between objects, but also expands on it.

Harel and Tall (1991) treat abstraction as part of the process of generalisation and concept building. This part of the process seems problematic in the case of the function, where there is evidence of students using all kinds of erroneous schema which they have abstracted from the examples that have been presented to them. They suggest use of generic examples, which I will discuss at greater length below, as a means of assisting students in making abstractions and building concepts around formal definitions. I will refer to this work again later in this chapter in my discussion of generic examples.

Mitchelmore (1994) expresses the view that teachers' attempts to facilitate acquisition of concepts by abstraction often result in the formation of what he terms 'abstract-apart' concepts. 'Abstract-apart' concepts contain a few situations, all of a similar kind, so that when the student encounters new embodiments of the concept, they are not linked to the original ones. On the other hand, 'abstract-general' concepts already contain many different expressions of the abstraction and can therefore accommodate the new embodiments. This view is in contrast to the notion of a paradigm or generic example which has the concept grasped in terms of only one essential example.

Traditional theories of concept formation by abstraction are one attempt to characterise the relationship between particular and general in the learning of mathematics. Finlow-Bates and Eade have each looked at this relationship in the context of teaching and learning approaches to a particular part of the mathematics syllabus.

Finlow-Bates (1994) reveals some of the thinking of a few mathematics undergraduates on the relationship between particular and general in his study of their ideas of proof. A series of examples was thought to constitute proof of the closure of a set, whereas an informal proof was seen as an 'explanation'. Their responses seemed to indicate that they saw the informal proof as a suitable introduction to the examples rather than as a summary. In this sense they were exhibiting a preference for moving from general to particular, rather than using the particular as a launching point for generalisation. Finlow-Bates suggests that the way in which text books frequently set out a new technique, by describing it in general terms and following this explanation by examples of the technique in use, might contribute to this preference.

Eade (1993) sets out six different approaches to teaching about the area of a parallelogram which differ in their intentions concerning particular and general. These approaches range from giving examples of a formula in use without justification and then setting some similar exercise for practise, to providing some particular parallelograms and their areas with no explanation

and asking students to generalise. A further approach gives some particular examples of shapes whose areas can be worked out by a method analogous to that for the area of a parallelogram and asks students to work out the area of each, then to generalise. This last approach has the method for working out the area of a parallelogram both as a generalisation of calculations for particular parallelograms and as a particular example of a method for working out the area of a more general class of shapes. These various teaching approaches might be chosen by teachers as a result of different beliefs about the nature of students' acts of generalisation and about the purpose of learning to find the areas of parallelograms.

My reading of these works leads me to draw a distinction between concept formation based on abstraction from numerous examples and that based on a generic or single example (Krutetskii (1976) speaks of generalisations based on a single example as 'on-the-spot' generalisations). This alerts me to look for evidence of these types of generalisation made by students, and for my own and other teachers' expectations that such generalisation will take place. In chapter five I discuss types of generalisation and the usefulness of distinctions which I might make between them.

Generic Examples

In rejecting the classical abstraction model of concept formation, Freudenthal (1978) states his preference for a teaching method which employs 'paradigms'. A paradigm he describes as 'one example, which evokes the general idea' (p170) or the one necessary example. In the context of learning Latin 'amo' as an example of a first conjugation verb is a paradigm. It acts as a paradigm even though the transposing to other first conjugation verbs may be unconscious.

The notion of an example which is seen in some way as representing a generality has been taken on by a number of authors, often using the term 'generic example'.

Mason and Pimm (1984) discuss generic examples in a variety of contexts, suggesting $f(x) = |x|$ as a generic example of a continuous but non-differentiable function, $2/3$ as generic example of the set $\{2t/3t : t \in \mathbb{Z}\}$, and Kleenex as a generic example of a tissue. They point out that the role of an example is to help students to see the generality which is represented by the particular. In other words students need to see the examples as 'examples of' some more general statement.

Mason (1993) again points out that the teacher's experience of 'examplehood' when presenting an example to students may be quite different from the

students' experience. For the teacher it is an example of something whereas for the student the example is a totality.

I will relate this idea to my own experience in chapter five.

Harel and Tall (op. cit.) emphasise the generic example as a means of generalisation for students. They speak of 'generic abstraction' as the process of forming a new concept by consideration of one paradigmatic or canonical example. They suggest three principles for selecting effective generic examples:

The *entification* principle says that the context from which the new object's properties are to be abstracted must be familiar, i.e. the elements of the model must be conceptual entities for the students e.g. 'vector space' is more easily conceived of in terms of line segments than polynomials because the latter are conceived of as processes not objects. The *necessity* principle states that students must be able to see the reason for the abstraction they are being asked to make. The *parallel* principle says that the generic example must be treated in a way which can be paralleled later in the general case.

This last principle perhaps misses the point that it is the student's treatment of the example which is crucial. 'Irrelevant' properties of the example may continue to form part of the student's concept image. This does not, however, remove the onus on the teacher to keep stressing the generic features of the example and 'ignoring' other features.

Balacheff (1988a) uses the notion of generic example in the context of students writing proofs. Of his four categories of proof, generic example is the third and is characterised as follows: 'The generic example involves making explicit the reasons for the truth of an assertion by means of operation or transformations on an object that is not there in its own right, but as a characteristic representative of the class' (p219) His doctoral thesis (Balacheff 1988b) (pages 124 to 130) gives several instances from students' work of the use of such generic examples. He suggests that such a 'proof' is a step on the way to the formal 'thought experiment'.

The idea of an informal proof based on an example which represents the general is also mentioned in Semadeni (1984). This article suggests that 'action proofs' can be a way of making justification accessible to primary school children. An 'action proof' is an active, manipulative, generic justification.

Rowland (1996), in chapter 5 'Hedges', gives examples of children's proofs by generic example. The task is to find the number of ways in which ten can be expressed as the sum of two positive integers, and then to generalise the question to take in other totals. The children's use of specific examples to

illustrate how they counted the number of ways is seen as generic proof of the general case.

Each of these three authors sees 'generic proof' as a stage leading to formal proof. They differ in how close to formal proof they consider 'generic proof' to be.

Hazzan (1994) and MacHale (1980) draw attention to some of the dangers of over-reliance on canonical or generic examples. Hazzan made a study of students' understanding of group theory and in particular their ability to solve the equation $x = x^{-1}$ in the context of a group. Many of the students claimed that the only possible solution to this equation was $x = e$. One of the author's suggested explanations for this is that the students are relying on multiplication on the real numbers as their canonical example of a group operation, so that they assume that the only element which is self-inverse is the identity element. Features of this canonical example, which are not a part of the generality it represents, have been imputed to that generality. Hazzan links this over-use of the canonical example with the role of metaphor in understanding abstract concepts. The students see the group operation *as* multiplication, rather than *like* multiplication, so that one student says 'Suddenly, everything (in Abstract Algebra) looks so strange. I mean why isn't $a*b$ equal to $b*a$?' (p53)

MacHale regrets the fact that text book authors are so consistent in their counter-examples, so that, for instance, $f(x) = |x|$ is almost the *only* example to be found of a continuous but non-differentiable function. The use of a single counter-example supports 'monster-barring' (Lakatos 1976) that is it allows students to dismiss the counter-example and maintain their belief that, for example, all continuous functions are differentiable. In addition it does not allow students to locate what it is that is similar about a number of examples and that makes them representative of the general. This amounts to an argument against the use of single examples, and therefore against the use of generic examples.

The generic example then is seen as a stage between particular and general. It has been advocated as a teaching approach and observed as a stage in understanding. To see generic understanding as a stepping stone between particular and general is to deny the universality of the 'abstraction from many particulars' model of concept formation, since generic understanding removes the need for abstraction from a large number of examples.

This discussion of the role of examples in the formation of concepts alerts me to look for the teachers' and students' use of examples in their work on second variable problems. Are there instances of the use of generic examples, and if so

in what context? Is there evidence of students using examples as a basis for abstraction? Are there conflicts between the teachers' and students' conceptions of the role of examples? My discussion of these points is contained in chapter five.

Frameworks for Studying the Learning of Algebra

Although research on students' work with second variables is quite sparse, there is a great deal of work on the learning of algebra in general. In this literature there are a number of identifiable frameworks for describing how students develop an understanding of algebra. In this section I consider some of the more commonly used frameworks and how they have informed my own study. They serve to place second variable problems in a spectrum of algebraic development and to offer potential analogies between first encounters with variables *per se* and with second variables.

Arithmetic and Algebraic Thinking

One such framework makes a distinction between arithmetic and algebraic thinking. A number of studies have attempted to characterise the two modes of thinking and identify them in students' work. These studies agree that the student's mode of thinking is determined by the learner rather than by the problem, and that certain problems can be approached in a variety of ways, some of which could be described as arithmetic and some algebraic.

For example Schmidt (1993) describes the work done by pre-service teachers on a series of eight problems. Some approached all of the problems in ways which the author identified as arithmetic, some used only approaches identified as algebraic and some used a mixture. The method chosen was as much dependent on the preference of the teacher as on the nature of the problem.

Lins (1992) looked at students' approaches to solving word problems and labelled as 'arithmetic' those that saw the problems in terms of quantities and their arithmetical relationships. Other approaches were subdivided into 'internal' and 'analytic', as types of algebraic thinking. 'Internal' thinking performs operations according to theories of algebra without referring out to any non-algebraic model (e.g. scale-pans). 'Analytic' thinking takes the unknown and treats it as known in order to discover its value. His study was to establish the adequacy of this categorisation of students' solution methods.

The introduction of analytic thinking was described graphically by Mary Boole (quoted in Tahta 1972) as follows:

'In this problem, besides the numbers which we do know, there is one which we do not know, and which we want to know. Instead of guessing whether

we are to call it nine, or seven, or a hundred and twenty, or a thousand and fifty, let us agree to call it x , and let us always remember that x stands for the Unknown.' (p55)

This description makes very clear both the way in which the analytic method is algebraic, and the way in which it builds on arithmetic thinking.

In Filloy (1984) the author describes the move from solving equations of the form $ax \pm b = c$ to solving equations of the form $ax \pm b = cx \pm d$ as a move from arithmetic to algebraic thinking. Whereas the former type of equation might be solved by reversing arithmetic operations, solution of the latter requires operating on the unknown. This move is also paralleled with a historical development in algebra as described below. Filloy (1985) reports on an attempt to teach the solution of such equations by using geometric models. The attempt was not considered particularly successful.

A major concern of authors has been the possibility of helping students to move from arithmetic to algebraic thinking. Meissner (1979) observed that children used a guess-and-test procedure rather than inverse operations to solve questions such as $? \times 4 = 24$ and $? + 9 = 15$. He developed an approach using the 'One-Way Principle' which initially avoided any use of inverse functions in solving problems. It encouraged understanding of the 'forwards' procedure by suggesting trial and improvement procedures and calculator games. The intention was to teach algebraic manipulation after the procedure had been understood.

This approach was revisited by Meissner and Müller-Philipp (1993). In the study reported here students were helped to link equations with graphs without using any transformation of equations. The argument is that students find it easier to use guess-and-test procedures than inverse operations because it is not necessary to make the 'formula' explicit. A first step in inverting a formula is to create an expression for it. 'Guess and test' can be used without being explicit about the process. In the language of other authors this amounts to suggesting that students should be encouraged to use arithmetic rather than algebraic thinking for certain classes of problems. It acknowledges that algebraic modes of thought come later in a student's development and recommends capitalising on the opportunities offered by technological advances to use arithmetic approaches to problems which have traditionally been used to develop algebraic thinking.

In contrast, Kieran et al (1993) report a study involving pupils in solving problems of the type 'Karen receives \$20 base salary plus \$4 for each subscription she sells. How many subscriptions must she sell to earn at least

\$50?'. A specifically designed software suite (CARAPACE) was used to allow students to set up procedures and apply them repeatedly to different inputs in a 'guess and test' procedure. The authors report that students tended to fall back on inverse arithmetical methods to solve problems even when they had been introduced to the computer procedure.

The 'inverse operations' method of solving equations of the form $ax \pm b = c$ can be seen, in different contexts, as arithmetic or as algebraic. It is arithmetic because it begins from the known and works towards the unknown, in contrast to Lins' 'analytic' approach. It is algebraic because it deals with operations on operations (the inversion of arithmetic operations) rather than using a trial and improvement approach which makes repetitions of the same sequence of operations.

Sutherland and Rojano (1993), whilst adopting a similar teaching approach to that used by Kieran, justify it very differently. In this study pupils worked on word problems using a spreadsheet to operate a 'guess-and-test' procedure. They claim that a common assumption is that symbolism is the last stage in understanding algebra. However Vygotsky's theory is that use of symbolic language can aid the development of understanding. Hence using computer packages which demand symbolisation can assist the formation of the function concept (for example) rather than necessarily following it. In Sutherland (1991) evidence that pupils learn through the use of symbols rather than by translating what they already know into symbols is presented in two forms: the ability to construct a program in Logo or on a spreadsheet without planning on paper first; the use of 'variable' language in speaking about what they have been doing.

Algebraic thinking, then, can be seen to be characterised by the analytic method, by the use of inverse operations, by the use of formal language or by the treatment of algebraic expressions as objects rather than processes. The contrasting lines of argument seem to be on the one hand that arithmetic thinking is the natural mode and should be encouraged as long as it is sufficient to solve the problem, perhaps with the aid of modern technology, and on the other hand that algebraic thinking is the more versatile approach and should be encouraged even where arithmetic methods would be sufficient to solve the problem. These two arguments perhaps stem from different views on the purpose of solving word problems. One view sees them as an end in themselves, whilst another sees them as a means of introducing algebraic methods.

Radford's argument (1995) is different again from these two. In an analysis of algebra of one unknown in medieval Italy, he points out that the symbolism used in this kind of algebra was developed in order to solve new kinds of problems. This analysis leads him to suggest that, in teaching algebra of one unknown the symbolic language be developed in response to new types of problem to be solved, rather than as an end in itself.

Ideas on the meaning of algebraic thinking, then, are various. To some extent the meanings reflect the type of problem which is of interest to the researcher. Would it be useful to arrive at a meaning of 'parametric thinking' along the same kind of lines which would be relevant in the context of second variable problems? This has perhaps been attempted by those who speak of 'functional algebra' as a stage in understanding (see next section). My own summary of the mathematical awarenesses I have identified as being relevant to second variable situations appears in chapter nine.

Historical parallels

Many of the theoretical frameworks used by researchers as a basis for their study of the learning of algebra are drawn from an analysis of the historical development of algebra. In particular much reference is made to Diophantus' development of manipulation of the unknown and to Vieta's use of letters to stand for coefficients in the polynomial equations he was solving.

Diophantus was remarkable for having introduced a limited use of symbols into Greek mathematics, which previously had expressed itself as an argument written entirely in words. He used symbols for the unknown and for its square, cube, reciprocal etc. However he appears never to have used more than one unknown.

The advances achieved by Diophantus are the subject of a chapter by Radford (1996). He shows how a problem which in modern times we would think of as algebraic was tackled by the Babylonians by drawing on geometric understanding. The work of Diophantus contributed the beginnings of deductive reasoning to make links between propositions concerning numbers, rather than using methods to calculate unknown numbers. Among the questions which the author raises as a result of his analysis is one concerning the potential usefulness of cut-and-paste geometrical procedures to awaken students' analytical thinking. For example, he describes a geometrical interpretation of a process similar to 'completing the square' which was used to solve problems of the form 'Find two numbers such that their sum and their product equal the given numbers'.

Vieta's most celebrated contribution was the use of different types of letter to stand for givens, as well as the, by then more conventional, use of letters to stand for unknowns. His use of consonants for given quantities distinguished them from unknowns which were represented by vowels. Most importantly it allowed the general expression in symbolic form of a method for solving an equation because it allowed a distinction to be made, in the interpretation into symbols, between given and unknown.

Kieran (1994) maps the historical development of algebra under the headings:

1. operational algebra
2. algebra of a fixed value
3. functional algebra.

The first of these could also be described as process-oriented, the second as algebra of an unknown and the third as the result of the move made possible by Vieta.

She parallels these stages with developments in an individual's understanding. At the three stages they would become able to solve problems of these types respectively:

1. Single appearance/reversal problems e.g. 'Amy has 5 more marbles than Bill and Bill has twice as many marbles as Ken; if Amy has 49 marbles how many does Ken have?'
2. Interpreting '15 more than $x - 30$ ', solving $3x + 5 = 2x + 12$
3. 'Which is larger $2n$ or $n + 2$? (Küchemann 1981), problems involving parameters (APU (Assessment of Performance Unit), 1980)

Rojano (1994) describes the move from stage one to stage two as a 'didactical cut'. It is exemplified by the move from equations in which the unknown appears only once, on one side of the equation, to linear equations where the unknown appears twice. She describes the results of an experiment concerning the equation $2x + 3 = 5x$, where children substituted a constant for the $5x$, and the equation $x + 5 = x + x$, where a typical response was 'this x (the one on the right) has a value of 5 and the other two (one on each side of the equation) can have any value (the same value for both)'. Kieran also addresses this issue in Kieran (1981) where she reports a teaching experiment which attempted to overcome students' difficulties with equations of this type. This paper focuses on interpretations of the equals sign as an explanation of students' difficulties. The 'do something' interpretation of the equals sign is unhelpful in equations where the unknown appears on both sides.

By contrast, however, Pirie (1995) suggests that the 'didactical cut' is a feature of a particular teaching approach and not a necessary consequence of the mathematics. She describes one teacher's method for teaching the solving of $ax \pm b = cx \pm d$ to 'low ability' pupils. The report focuses on the success of two pupils who arrived at their own formulation of a general method for solving these equations after they had worked through a series of judiciously chosen examples. An important feature of the series of examples is that it starts with an equation where the structure is simple but the numbers involved are not so small that the calculation required can be done unconsciously. (The first equation is $\square + \square + 18 = \square + 53$).

Although Pirie stresses the teacher's positive approach to his pupils' abilities as part of the explanation for his pupils' success (he believes that they can do complicated mathematics and calls them 'mathematicians') she also points to his treatment of the equation as a statement of fact rather than an invitation to begin a process. In this sense she acknowledges the importance of understanding the equals sign as a symbol of equality rather than as an alternative to 'makes'.

I would also point to this teacher's sequencing of problems as a factor in the pupils' success. His first equation ($\square + \square + 18 = \square + 53$) is simple in that it only requires one operation to directly calculate the value of the unknown (as opposed to, say, $\square + \square + \square + 17 = \square + 53$) and in that it requires that only one box is 'ignored' on each side in order to focus on $\square + 18 = 53$ (as opposed to, say, $\square + \square + \square + 18 = \square + \square + 53$). However it is not simple in that it uses small numbers (as, for instance, $\square + \square + 2 = \square + 5$). The larger numbers force the pupils to use addition consciously because they need to press the '+' key on the calculator and allow them the opportunity to develop the notion of using the inverse operation of subtraction to streamline their solutions (they do this in a later lesson).

Harper (1987) speaks of both of the historical developments (the Diophantine development of working directly on the unknown, and the Vietan development of making a symbolic distinction between unknown and given) as being relevant to his observations of the work of students on the problem 'If you are given the sum and difference of any two numbers show that you can always find out what the numbers are.' He distinguishes three categories (rhetorical, Diophantine, and Vietan) of ('correct') responses to this problem from pupils aged 11 to 17. He describes these developments in algebraic thought as being from *rhetorical* to *syncopated* and *syncopated* to *symbolic* respectively (after Boyer 1968). An example of a rhetorical method comes from a 12 year old:

'You divide the sum by 2 then divide the difference by 2. Then to get the first number add the sum divided by 2 to the difference divided by 2. To get the second number take the difference divided by 2 away from the sum divided by 2.

$$\begin{array}{ll} \text{e.g. sum} = 8 & \text{difference} = 2 \\ \frac{8}{2} = 4 & \frac{2}{2} = 1 \end{array}$$

$$\text{1st number} = 4 + 1 = 5$$

$$\text{2nd number} = 4 - 1 = 3'$$

An example of a Diophantine solution is produced by a thirteen year old

$$'x - y = 2 \quad (1)$$

$$x + y = 8 \quad (2)$$

$$(1) + (2) \quad 2x = 10$$

$$x = 5$$

Substitute into (2)

$$5 + y = 8$$

$$y = 8 - 5$$

$$y = 3$$

You can do this for any numbers'

A Vietan solution would take the form:

$$n = x + y$$

$$m = x - y$$

Add together $m + n = 2x$

Find x and substitute back for y

One of Harper's purposes in this study was to examine the idea that ontogenetic development of mathematical concepts parallels phylogenetic evolution. He concludes that pupils do acquire the concepts in the phylogenetic order.

Both Kieran (op. cit.) and Harper go on to make suggestions about the way in which teaching designed to move pupils on into the last of the three stages can be improved. Harper recommends making the difference between 'unknown' and 'given' more explicit in text books. Kieran describes an experiment which attempted to start with functional algebra rather than going through the 'algebra of a fixed value' stage.

Sfard and Linchevski (1994) describe the historical development of algebra under the headings:

1. algebra as generalised arithmetic:
 - (a) the operational phase

(b) the structural phase:

(i) algebra of a fixed value (an unknown)

(ii) functional algebra (of a variable)

2. abstract algebra - algebra of formal operations and abstract structures.

Stages 1(a), 1(b)(i) and 1(b)(ii) correspond broadly to the three stages identified by Kieran and by Harper.

Sfard and Linchevski are also concerned with the assumption of a parallel between historical and individual development in mathematical understanding and examine the difference between logical, ontological and psychological development of mathematics.

There is a remarkable degree of agreement between these authors on the historical stages in the development of algebra and on its parallel in psychological development, although some have queried whether this order of development is a feature of our teaching tradition rather than a necessary consequence of the nature of the human mind.

My concern is with the move from the second to third stage. This historical background has provided me with a framework for thinking about the distinctions between algebra of one variable (the solution of equations in one unknown and functions of one variable) and algebra of more than one variable (solution of equations with coefficients expressed in terms of literal symbols, families of functions). Is it appropriate to equate my students' first encounters with second variable problems with this move as described by these authors? How convincingly does such a theory explain what I observe in the behaviour of my students? Does it help me to formulate alternative teacher actions which might offer new awarenesses to the students? I will return to these questions in chapter nine.

Process/Object Distinctions

Fundamental to much recent and current research on teaching and learning algebra is the special ambiguity of mathematical concepts. This ambiguity can be characterised as process/object. For example the notation $2x + 3$ can be understood to describe a process 'double a number and add three' or an object, the result of applying the process to x . Similarly a polynomial may be thought of as a process with input and output or as an object which may be manipulated or taken as an element of a set. The process/object ambiguity was one of several pointed out by Freudenthal (1983) in his chapter on The Algebraic Language.

Aspects of this distinction have been approached by a large number of authors, as I describe below.

Tall's was one of the early inputs into the process/object debate. His term 'procept' is designed to convey a combination of process and concept. He describes the meaning in Tall (1991) as process and product represented by the same notation, for example $3 + 5$ or $2x + 3$. Manipulation involving ' $2x + 3$ ' is impossible if it is seen only as a process 'double a number and add three' and not as the result of the process, or product. He suggests that appropriate use of a computer enables students to focus on products by doing the process for them.

Harel and Kaput (1991) use the term 'object-valued operators' to describe a class of operators which they exemplify by 'parametric functions such as $f(x) = ax + b$, $f(x) = \sin(ax)$, $f(x) = \log_a x$ ' (p87). The operator described as object-valued in this case is the correspondence between parameter (a, b here) and function. Other examples are given from the fields of linear algebra and group theory. They quote Harel (1985) as finding that 'students usually had difficulty dealing with such a correspondence, unless they were able to tag the outputs of the correspondence with familiar geometric figures, such as lines or planes' (p87). By analogy this work suggests that students might find it easier to understand parametric functions, examples of which are given above, as objects if they understood them in the context of their graphs.

Gray (1993) states that, 'The invention of symbolism provided mathematicians with the means of representing the process/object ambiguity' (p2). He suggests that the ability to exploit this ambiguity is the requirement for success in algebra.

Sfard's writing on the process/object distinction uses the terms 'operational' and 'structural' to describe mathematical conceptions and forms of understanding. Sfard (1991) contains an argument on the basis of history and cognitive schema theory that operational understanding usually precedes structural understanding (i.e. process precedes object).

However Dubinsky and Harel (1992) describe a study whose objective was to distinguish between behaviours demonstrating an action conception of function and a process conception. The route from one to another seemed complicated by other factors and a linear measurement of progress along it was deemed impossible.

Kieran (1992) includes in her chapter on research into the teaching and learning of school algebra a section on 'Psychological Considerations'. She refers to Sfard's notions of *structural* and *operational* concepts and the three stages she identifies in concept development. These are interiorisation, when a process is performed on some familiar mathematical objects; condensation, when the

process becomes more familiar and condensed; reification, when the student suddenly sees process as object.

Kieran claims that development in algebra can be seen as a series of process-object transformations. Students have to conceive expressions as objects not processes to be able to understand (for instance) that $2a + 2b$ is the same as $2(a + b)$. Another example is translating word problems into algebraic equations. The move from arithmetic to algebraic thinking about these problems is analogous with a move from procedural (she uses this term as equivalent to Sfard's 'operational') to structural. She points out that students asked to work on this kind of problem by writing instructions to a computer are being invited to work procedurally. This sets her view in contrast to one which has algebraic thinking characterised by formal language or by use of inverse operations.

Dubinsky (1991) uses the word 'encapsulation' to describe the conversion of a process into an object. He sees this as one type of construction alongside extensional generalisation which is equivalent to widening the applicability of a schema. Pimm (1995) contains some harsh criticism of this chapter on the grounds of sloppy thinking and writing. The main criticism of 'encapsulation' as an idea is that the claims made for it are too general. The ubiquity of encapsulation is not established but assumed.

Dubinsky's ideas on encapsulation are bound up with a rejection of the strictures of classical abstraction as a framework for describing concept development. On p121 he says 'We agree with Tall (1986).....that understanding mathematical ideas comes from sources other than looking at many examples and "abstracting their common features".....'. Rather he starts from Piaget's term 'reflective abstraction'.

Piaget (1971) says of reflective abstraction that it 'does not derive properties from *things* but from our ways of *acting on things*, the operations we perform on them.' p24. He gives the genesis of the idea of 'group', which arose in the context of operations on functions, as an example of reflective abstraction. Another example of reflective abstraction is to shift attention from objects to transformations of objects for example, the symmetries of a square. Having thus created new objects we can repeat the process by considering transformations of our new objects.

Dubinsky describes reflective abstraction as drawing properties from mental or physical actions. This involves consciousness of the actions. He also speaks of it as constructing new combinations by a conjunction of abstractions and says that these construction aspects are 'the essence of mathematical development'.

Bednarz (1993) interprets the process/object distinction in the context of solving arithmetic problems. She identifies some arithmetic or pre-algebra strategies for solving problems of the type 'Jane has 3 times as many marbles as Katie and Katie has 5 more than Sue. Altogether they have 70 marbles. How many does each girl have?' In students' reaction to being shown an algebraic approach she notes the typical desire to break the problem down into stages and a reluctance to operate on the unknown, particularly in the case of a composite operation (For example in the above Sue has x , Katie has $x+5$, Jane has $3(x+5)$. The last expression involves a composite operation on an unknown). She points to students' unwillingness to accept composite operations on unknowns as an example of their operating at the level of process rather than object.

Vergnaud (1991) uses the language of 'concepts-outil' and 'concepts-objets' to describe a similar but not identical distinction. The transformation from concepts-outil to concepts-objets allows a propositional function to become an 'argument' i.e. something to be operated upon. His emphasis in this paper is on bringing to consciousness the processes performed by the student so that this transformation is possible.

The distinction he is making is concerned mainly with the pupil's level of awareness of the concepts they are using, rather than their ability to operate on the same concepts as objects.

He describes an approach which involves using language and notation to make procedures more explicit or to draw pupils' attention to generalities behind the particular problems they are working on. The particular class of problems he used is exemplified by this question

'Melanie goes to buy a cake from the baker. She has paid him 8 francs. She counts what she has left in her purse and finds she has 7 francs. She wonders if she has lost some money and wants to know how much money she had before buying the cake.' (p160, my translation)

He suggests that the initial amount of money held by Melanie could be referred to in several ways:

- by using the imperfect tense¹ and a subordinate clause 'How much money did Melanie have before she bought the cake?';
- by using a pronoun, a complement and an adverb 'what she had before';

¹ The French text "*Combien Mélanie avait-elle avant d'acheter ce gâteau?*" has the verb in the imperfect tense, though the English translation does not.

- by speaking directly of the 'initial state' or 'point of departure'.

These three expressions represent a gradual shift from language specific to this problem to language which points to a generalisable feature of this problem i.e. the initial state.

This example illustrates the technique of drawing the procedure involved to the attention of the pupil by using language which is not particular to the problem but is transferable to a similar problem. Continued use of such language would contribute to a process of enculturation; explicit reference to the language could assist some students to shift their attention from the specific problem to the type of problem. This could be a first step in seeing the constituent parts of this type of problem (the initial state, the operation and the final state) as objects.

The process-object framework also carries various emphases in its various forms: process/object, procept, operational/structural, reification, encapsulation. My concern with this body of work is in its usefulness in describing students' behaviour and in suggesting alternative teacher actions. I have found the process/object distinction useful in interpreting students' work on graphs and have described this interpretation in chapter six.

The contribution of this chapter to my work is to suggest a variety of frameworks through which to view students' working on second variable situations. In later chapters I call on these frameworks to help me to understand, in the sense of giving me a metaphor for, classroom incidents and my experiences in working on mathematics.

In chapter nine I return to these frameworks and discuss in more general terms their usefulness to me in understanding my experiences.

Chapter 3 Methodology

At the end of chapter one I set out the following questions as those with which I began my study.

What are students' experiences of facing second variable situations?

What mathematical awarenesses are available through these situations?

What techniques are available to teachers for working with students on these awarenesses?

What is involved in shifting attention from particular to general and from general to particular?

My survey of literature, which is described in chapter two, provided a number of perspectives from which to look for answers to these questions. These were

- (1) the inadequacy and ambiguity of language used to describe second variable situations
- (2) 'correct ordering of quantifiers' as a criterion for judging understanding of the roles of parameters
- (3) the potential for developing a graphical understanding of the meaning of parameter, in particular through the use of graphing technology
- (4) the role of examples in classical abstraction and in generic abstraction
- (5) historical parallels to the development of students' thinking
- (6) process/object distinctions and ambiguities in algebraic language.

In this chapter I explain my reasons for choosing particular methods in my search for knowledge. This explanation begins with broad issues of world

views and research paradigms, and moves through factors involved in the particular nature of my area of interest to a closer description of the elements of my chosen method. Starting from a very general look at what constitutes knowledge in the area of mathematics education, I gradually focus down on the particular concerns of my enquiry.

What is Knowledge in Mathematics Education?

The Nature of Knowledge in Mathematics Education

Before I can make any claim to have augmented the body of knowledge in mathematics education, I need to explore the nature of that knowledge. The term 'mathematics education' expresses a field of enquiry, or a set of human situations (for example students studying mathematics in school) rather than a discipline, in the sense of a set of criteria for judging the validity of a claim to knowledge. One way to proceed, then, is to draw analogies between the field of mathematics education and other fields of enquiry. The kind of knowledge I try to look for will depend on which other fields of enquiry I am drawing an analogy with. If I work by analogy with the scientific paradigm I might ask 'what is a fact in mathematics education?'. If I work by analogy with literary criticism I might ask 'what is an original interpretation in mathematics education?'. Or by analogy with epidemiology I might ask 'what is a condition or syndrome in mathematics education whose high risk factors I might identify?'. Allied to each of these questions is one about the methods by which these forms of knowledge are established.

Scientific Facts

For purposes of comparison here I choose to mean by 'fact' a claim which is no longer seriously challenged by members of the scientific community. In the scientific community claims predominantly acquire the status of 'fact' by being validated by empirical evidence. In addition, their domain of validity or level of generality must be sufficient to render them worthy of acknowledgement.

As 'facts' in mathematics education I offer

'not every student in every lesson learns what the teacher intended them to learn'

'a student who can satisfactorily answer a question today may not do so next week'

'the context in which a mathematical task is set can have a great effect on a student's degree of success with that task'

A number of characteristics of these 'facts' are worth noting. First, all three are logically weak statements, because they are denials of universals (these universals being 'every student in every lesson learns what the teacher intended', 'any student who can answer a question satisfactorily at any time will remain able to do so for all time', 'the success of any student is constant across a range of tasks all addressing the same mathematical principle'). In order to find a 'fact' which I considered undeniable I have had to choose weak statements.

Secondly, in the operation of an education system, we, that is myself and other teachers, act to a greater or lesser extent as though each of these universals were true. So, for instance, we set out a curriculum which the teacher will teach in the expectation that students will learn it. We measure students' achievement by their performance on a test taken on a single day. We make judgements about whether or not students can, say, solve simultaneous equations, on the basis of a single examination question, regardless of context. Many aspects of the structure of schooling appear to confirm a belief in the universals. However, operating within this structure is not necessarily a denial of the contrary 'facts'. A multitude of factors compel systems and individuals to act as though the universals were true.

Thirdly, we quickly become convinced of the falsehood of these universals by experience. For most, the experience of their own schooling is sufficient for them to deny their truth. But the extent of the falsehood, or the proportion of counter-examples takes longer to appreciate. Each represents a significant area of learning to be approached by a novice teacher.

(The last three paragraphs contain a large number of unsubstantiated statements which contribute to my argument. I expect that you have had little pause before accepting these statements because they fit with your experience of learners of mathematics and novice teachers. My own belief in the statement 'Each represents a significant area of learning to be approached by a novice teacher' is firmly attached to an experience from my time as an initial teacher training student. In marking a test which I had set for a class I had taught for a few weeks I was taken by surprise and greatly upset by the extent to which the students had not learnt what I thought I had taught them. Before this experience I would have given intellectual acknowledgement to the proposition that not all students learn what I intend to teach them. Afterwards my knowledge had a different quality. I mention this experience because it may bring to your mind similar experiences of your own, or reported to you by novice teachers. I contend that such confirmation of

assertions by comparison with your own experience is a common and useful response).

I conclude that facts in mathematics education differ from facts in the realm of scientific knowledge in the following respects. First, in order to achieve wide acceptance they must be logically weak. Secondly, in many cases their denial is embodied in a naive model of teaching and learning which informs the operation of an education system. Thirdly, appreciating the truth of these facts must be a matter of experience and not just intellectual acknowledgement.

Interpretation

In interpreting a passage or work of literature a critic may draw on her knowledge of the author's life and other works, of the period in which it was written and especially the ideas and metaphors which would have been current in the author's society. She also compares this author with others. She then accounts for, or explains, the work under consideration and the intention of the author.

In mathematics education I might make an interpretation of the work of a student by making some considerations parallel to those described above. These would include as much information as possible about the student's learning experiences, about her previous work and about the circumstances in which this piece of work was done. My interpretation would also make comparisons with other students, or with my model of a typical student's understanding, in trying to account for, or explain, the student's intention.

There are important differences between interpretation made by a literary critic and by a researcher in mathematics education. First, for the critic, the piece of writing is the principal object of study. The question as to whether the critic's conclusions are valid for a large category of writings does not arise. The uniqueness of the author and her writing is paramount and the writer is a key figure for the critic and the reader. In contrast, a mathematics education researcher sees the student whose work they study as in some way representative of other students. This particular student is of interest to the reader only in so far as she resembles students in general or the reader's own students in particular. Such resemblance must remain largely a matter for the reader.

The validity of such interpretations must be approached differently in the two cases. In literary criticism the reader remains to be convinced by the critic's argument and by the fit between observable facts and the suggested interpretation. That is, the reader must judge the coherence and consistency of

the interpretation and its adequacy to account for a breadth and depth of factors. Part of the critic's argument might be that her interpretation is less implausible than any other. She may also claim to have illuminated the text in an original way.

In the case of the mathematics education researcher the above remains true, but in addition the reader must be convinced that the significance of the interpretation goes beyond the personality and circumstances of that student.

The model of interpretation has something to offer the mathematics education researcher in its emphasis on context, plausibility and illumination, but she needs to have a further concern for the domain of validity of her findings. The relevance of the interpretation must go beyond the particular student.

Large Scale Survey results

Survey results in the area of epidemiology are of the kind 'Smoking is associated with increased risk of heart disease' or 'Use of penicillin is associated with an allergic reaction in approximately ten per cent of patients'. Such claims can be made on the basis of statistical association, rather than by reference to any causal connection. The establishment of such a causal connection and the mechanism by which it occurs is a vital but separate enterprise. Similarly it is possible to make statements in social science of the form 'Fifty per cent of single mothers are on below average incomes'. Or in mathematics education we could find that 'This year seventy per cent of those beginning a course in 'A' level mathematics have achieved at least a grade B at GCSE'.

I wish to make three points about this last statement. First, the domain of validity of the statement is at the same time very large and very small. It, let us say, tells us something about the whole of the population of students taking 'A' level mathematics in England and Wales this year (though it is much more likely to be an estimate based on a sample which is taken to be representative). On the other hand it tells us little about last year or next year. It does not carry the status of scientific fact because its domain is not in any sense universal. Its currency extends neither beyond this year nor beyond this country.

Secondly, because it is a national summary statement it has little to say to me about the class I am teaching, except perhaps that they are typical or atypical in this respect. The kind of statement which could be supported by a large scale survey result is more likely to be of interest to educational policy makers, as an indicator of a trend, than to teachers (I recognise that these can be, and often are, the same people). This is because the kind of detail which might be of interest to me as a teacher is too complex to be measured in a replicable way across

hundreds or thousands of subjects. For example, I would be more concerned by some aspects of the experiences of pupils who embarked on an 'A' level course in mathematics with only a grade C at GCSE than by what proportion they form nationally.

Thirdly, few researchers would be happy to leave the statement as one of statistical fact. Their interest is in the factors which may combine to produce this phenomenon and in its implications. All that can be argued by the researcher is a fit with observations. Observations made by the researcher in other circumstances are likely to be just as influential on her explanation of the phenomenon as those made in the course of this study. She may argue, more or less consciously, from her reading of other studies in the area or her more general life experience. In addition, the reader will bring her own experiences of a similar or contrasting nature to her interpretation of the explanation.

Professional Knowledge

Although less well established as a field of enquiry than my previous three areas of knowledge, the way in which professionals operate has become a matter of popular interest and concern and is clearly highly relevant to the field of education. Professionals operate by using a 'body of knowledge' which does not consist only of well defined statements of fact, but also of guiding principles which are to be interpreted within the professional situation. Schön's account of the nature of professional knowledge (Schön 1983) gives examples of these guiding principles in action and explores ways in which this knowledge is acquired by new entrants and built on by experienced practitioners.

In mathematics teaching the following are examples of such guiding principles:

'the "moving the figures" approach to multiplying decimals is less likely to lead to mistakes than the "moving the decimal point" approach'

'an initial introduction to sine and cosine as functions rather than ratios aids students in later generalisations'

Although teachers might use these statements for guidance in preparing their teaching plans, they will interpret them in the context of their particular situation. Other factors to consider will include the students' prior understanding and individual learning styles.

In a similar way a doctor might decide which drug to prescribe for a patient based on his knowledge both of the actions of the available drugs and of the patient.

The establishment of such guiding principles does not come about by undergoing large scale teaching experiments, but by theorising and reflecting on experience. The practitioner draws on their knowledge of their subject and of their client, as well as their experience of the relationship between the two.

Having looked at comparisons between mathematics education research and other fields of enquiry I turn to ways in which educational researchers have conceived of knowledge and the search for knowledge in their own field.

Educational Research

Four Paradigms

I will begin with a brief characterisation of some partly contradictory positions within education research.

Guba (1990), arguing from the viewpoint of a constructivist, describes four educational research paradigms in the following way:

Positivism Reality has an objective existence independent of time and observer. It can be discovered by empirical enquiry through hypothesising and testing. Any acceptable theory is falsifiable.

Postpositivism Reality exists but cannot be fully comprehended. Completely objective enquiry is not possible but is an ideal to aim for.

Critical Theory Reality can only be viewed through a value window. The choice of a particular value system empowers and enfranchises some whilst disempowering others. Therefore enquiry becomes a political act.

Constructivism Realities exist in people's minds in the form of mental constructions and are dependent for form and content on those that hold them. Constructivist methodology is hermeneutic/dialectic. The hermeneutic aspect depicts the constructions of participants while the dialectic aspect compares them. 'The hermeneutic/dialectic methodology aims to produce as informed and sophisticated a construction ... as possible' (p26)

The relationships between these paradigms are well-rehearsed in the chapter quoted and elsewhere. My response is summed up in the remainder of this section. It is not my intention to give a comprehensive critique of each paradigm, but to explain why none fitted my beliefs and purposes.

Positivism is internally consistent (by some standard of rationality) but does not fit with my observations of how scientific or social scientific discovery takes place. In a social science context it must contend that only behaviour is observable. This makes it difficult to reconcile both with my area of interest, which is in students' awarenesses and experiences, and with my preferred

stance towards my identity as researcher, which includes my aim of personal professional development.

Postpositivism argues reality exists but our experiences of it are mediated by our senses and by our past experiences. So we create mental models of reality which can only ever be *models*. Their validity must be judged by their fit with our experience of reality, since it is not possible to compare them directly with reality. However we should strive to understand reality by enquiry which is as objective as possible. In other words, the education researcher is trying to make as good a description as possible of the reality which is happening in classrooms. In the realm of social interaction the level of description and analysis required for a study to be as objective as possible tends to place its conclusions outside the realm of usefulness, either by being too broad, or by being too detailed. For example, a conclusion that the use of calculators in the secondary school is associated with an improved understanding of decimals would be so broad as to be of no practical use as well as being open to dangerous misinterpretation. On the other hand a finding that students who used 'guess-and-check' procedures on the calculator to find square roots performed better on manual addition of numbers with one decimal place would be at the other extreme. (I do not mean to say that a study which included these among its findings would be worthless. The researchers would no doubt make an interpretation of these findings which drew on their experience of mathematics teaching both before and during the study. Such interpretations may find acceptance amongst readers.)

I suggest that the aspects of teaching and learning which are of interest to me are not best studied by being as objective as possible. 'Experiments' involving human subjects are not replicable and many educationally desirable outcomes are not quantifiable in any way which preserves value. Further, educational 'treatments' are not easily specified in a way which makes them objective (by which, in this context, I mean independent of the teacher). Understanding, however we construe it, can only be studied by very indirect means and through the medium of my own understanding.

In response to the critical theory paradigm I acknowledge that my view of reality is through a window defined by my values. However I am persuaded that the situations I am viewing, with the kind of interest I take in them, do not look very different through the eyes of somebody with very different political values from my own. I judge therefore that the political aspects of my enquiry are not the most important in terms of choosing an appropriate method.

Finally, constructivism seems to make the whole notion of research redundant. The metaphors of 'adding to a body of knowledge' or 'making an original contribution' have no meaning if contact is not possible between my reality and yours as reader.

A constructivist approach to research is based on a problematisation of communication and hence needs to address the issue of how a research community validates the work of its members. This process of validation must involve some form of communication of the researchers' constructions. Von Glasersfeld's assertion (von Glasersfeld 1987) that

'what determines the value of conceptual structures is their experiential adequacy, their goodness of *fit* with experience ..' (p5)

leaves the question of validity with the individual and her experience.

I find descriptions of constructivist methodology lacking in consideration of how the research community evaluates and responds to the work of its members.

Reflexivity

Winter, in his books on action research in education (Winter 1989, Winter 1987), draws on an academic tradition including the writings of Heidegger, Polanyi, Derrida and others to support the 'thesis of reflexivity'. This thesis claims that an observer's judgements are inevitably framed by the language which they associate with the situation observed and reflect their preconceptions and prior experiences.

Winter reaches the conclusion that small scale research should include a reflexive critique. The basic procedure of a reflexive critique will have three stages: (1) accounts will be collected (2) the reflexive basis for these accounts will be made explicit (3) claims may be transformed into questions and a range of possible alternatives will be suggested where previously particular interpretations have been taken for granted (Winter 1989 p43)

Although I concur with much of the argument that he sets out in arriving at this pattern, I differ from his suggested response to this argument. First I find the claim that 'the reflexive basis for these accounts will be made explicit' to be exaggerated. It is possible to examine factors which may have led to particular judgements being made as expressed in the account, but I dispute that it is possible to identify the reflexive basis for these accounts. To do so would be to claim an external objectivity whose denial is the very essence of the reflexive critique. Secondly the claim that 'previously particular interpretations have been taken for granted' needs examining. My question is 'taken for granted by

whom?'. If the person who previously took them for granted was the account giver, the judgement passer, and the person who challenges them is the researcher then we have merely a difference of opinion. If the person who previously took these interpretations for granted was the researcher, then it is difficult to see how the alternative interpretations might be arrived at. An interpretation which is taken for granted is, by definition, not questioned.

The use of his three stages is both too difficult and too easy. The questioning of genuinely 'taken-for-granted' interpretations cannot be done by formula and may not be voluntary. At the least it must be prepared for and worked on. On the other hand, to produce my own interpretations in contrast to those which I claim to be 'taken for granted', with no necessary suggestion that my interpretations are more valid or more useful, is both easy and relatively uninteresting.

My Choice of Research Process

My concern in this enquiry is to understand the experiences of students in facing second variable problems and to suggest ways in which teachers might act in order to share their mathematical awarenesses in this area. None of the research traditions which I have outlined above suits my purposes very closely. In order to understand the experiences of others I must draw on my own experiences in interpreting what they do and say, rather than objectifying and categorising their actions. In order to address mathematical awarenesses (my own and those of other teachers) I must work on and analyse mathematical tasks, rather than treating the mathematical context as a distraction. In order to be useful to teachers the new ways of acting must be accompanied by the awarenesses that suggested them, not simply taken on as prescribed actions.

The nature of the phenomena I wish to study determines the most useful approach to making the study and communicating its outcomes. My research questions developed through the interaction between my thinking on methods and my developing interest in the area of second variables.

The nature of some knowledge areas lends itself to precise language, that is, language which can produce high levels of apparent agreement as to meaning. In other areas much effort needs to go into negotiation of meaning in order for apparent agreement to be achieved, and this is done more by exchange of uses and of examples than by attempts at definition (for example, use of precedent in law). In such areas, 'knowledge' gained by construal of experience is most successfully communicated in the presence of similar experience. It is my contention, therefore, that researchers in such areas should, in attempting to

communicate the outcomes of their research, present an account of some of their experiences and call on the reader's recollections of similar experiences.

The area which I am studying, namely that of students' awareness of generality, is fraught with difficulty if I try to make inferences from data in any simple model. These awarenesses are by their nature difficult to put into words, especially for students, and I will continually need to use students' utterances and actions as 'circumstantial evidence' rather than proof of a particular awareness. For example, a student who was asked 'What is a quadratic?' replied that it 'has three bits to it'. I can use this incident to raise my own awareness of the visual impact of repeated exposure to stereotypical quadratic expressions, but I cannot make any definitive statement about the student's awareness of the nature of a quadratic. Equally there are inherent difficulties in locating a student's attention at any given time because any probe necessarily relocates it. As soon as I begin to question, for instance, whether the student's attention is on two as the highest power in a quadratic, I draw attention to this feature. I cannot seek answers to questions about students' attention and awarenesses by direct questioning.

So the crux of my research is to piece together a picture of my own understanding of students' awareness of a particular aspect of generality. This picture will be communicated by accounts and interpretations of these accounts. I will attempt to enable the reader to fill in details in the sketch by recalling their own experiences of teaching and learning mathematics.

Concerning objectivity, or the neutrality of the researcher, I have considered two, at least apparently, opposing models of research:

- formulate a theory which you test in an empirical way using neutral and objective test instruments (or as close as possible to this ideal in a social science setting)
- set out to gather data in a certain situation chosen for its intrinsic interest and formulate questions and/or theories as a result of reflecting on the evidence you gather

Neither seems satisfactory in the context of my interest because:

- as a teacher I wanted to take advantage, in gathering data, of my knowledge and experience of teaching students over many years, rather than positioning myself as a neutral test instrument.

- the nature of the phenomena which I wish to observe, especially the human interactions, makes particularly pertinent the warning that 'all observation is theory-laden'¹. The implications are that in making observations the researcher herself becomes the research instrument and cannot be neutral, and that it is not possible to produce theory grounded solely in data gathered for the purpose of the research, since the data that is gathered will necessarily depend on the observer's previously formed 'theories'.
- my interest is in communicating my awarenesses to others as well as in accruing knowledge. To that end I need to invite readers to trawl their own experience for similarities with or differences from what I am recalling.

I wish to employ a method which embraces the subjectivity of the researcher, rather than seeing it as an inadequacy, the effect of which needs to be minimised.

My model, then, is to start from my existing knowledge of my awarenesses gained from years of studying and teaching mathematics, to search for new insights in a systematic and disciplined way by

observing, recording, interpreting and communicating my experience of classroom events,

working on my own mathematics and making observations and records,

reading about the experiences and theories of others,

offering my accounts and/or interpretations to other teachers and accommodating to their responses.

My means of searching for new insights, that is through accounts, exercises and comparison of experience, at the same time offers a method of communication. It is through these same processes that I hope to engage the reader with this subject matter.

What is my research for?

My reading of other work in my area of interest gave me a number of frameworks for viewing this aspect of algebraic thought:

¹This misquotation apparently has its origins in Hanson (1958) which on page 19 says, "There is a sense, then, in which seeing is a 'theory-laden' undertaking. Observation of x is shaped by our prior knowledge of x ." The book which is quoted deals with philosophical aspects of elementary particle physics.

- the role of examples in concept formation
- arithmetic/algebraic thinking
- the Vietan 'stage' in historical development of mathematics
- the process-object framework

so that now I wanted to test these frameworks against my experience, to see if they helped me to explain my students' behaviour or helped me to consider alternative actions.

Thus the aim of my research is in the first instance to inform my own practice and, both as an aim in its own right and as an aspect of validation, to inform the practice of others. In addition I aim to make a contribution to the metaphorical debate sometimes known as the 'body of knowledge of the mathematics education community'. This contribution should be at the level of content and of method.

The informing of classroom practice is mainly concerned with teachers' decision-making. Decision-making concerning my classroom practice happens at a variety of levels. There are decisions to be made about curriculum and design of classroom activity which are mostly made before the lesson, are subject to many external constraints and may be beyond the control of the teacher. Within the lesson there are many small decisions to be made about how the class will run, for instance when exactly to call a halt to one activity and make a transition to something else. Then there are decisions about how to respond to or intervene in students' activity, for example in the following situations -

- I ask a question of the whole class and nobody offers to answer it
- an individual pupil comes to me for help on a particular problem
- walking around the class to look at pupils' work I find that two pupils have done something quite different from what I was expecting

It is this last kind of decision-making that I am most interested in. These decisions call on a teacher's knowledge of mathematics, of individuals in the class, of the psychology of learning mathematics as well as on their more general class management skills. They call for what Tripp (1993) describes as 'professional judgement'. Moreover they call for instant decision-making.

If I want to improve my decision-making skills I can work on my awareness of these various areas and I can focus on the decisions themselves.

In order to achieve this last aim I might work on two things. First there must be an identification of decision points which occur during or before a lesson. Secondly there must be an alternative action formulated.

My characterisation of the processes by which these things are achieved is borrowed from the Discipline of Noticing. The key components of this method are briefly explained in Davis (1990) and in Mason (1994). For my purposes here the important aspects are Recognising Choices, and Preparing-and-Noticing. On 'Recognising Choices' Davis says

'Decisions are about choices. By laying the strands of recent experience alongside the strands of past experience, you gain access to some possible ways of behaving in similar situations. You can choose to respond in a different way. You can also get ideas about alternative responses from reading other people's accounts, both practical and theoretical, and by watching colleagues in action, or listening to their description of Teaching Moments.' (p172)

and on 'Preparing and Noticing'

'In order to change your behaviour, you have to notice the opportunity in the moment. Yet at first you usually find that you only notice an opportunity after it has gone by. Eventually, with continued effort, the moment of noticing creeps forward, until you notice an opportunity and you take it, all in the twinkling of an eye.' (p173)

An important premise of my work is the need to sensitise myself to moments of decision. These are most usefully moments which I recognise as occurring frequently and in which perhaps I habitually respond in an unthinking way, by making an unconscious decision. I need also to recognise that there are alternative actions which I could take in these situations, and, finally, to prepare myself to take these actions by mentally entering a future moment and 'trying out' the alternative action.

However if this were all I aimed to do then my activities would be in the realm of personal professional development and not of research. In addition I want to expose and consider the theories which underpin my choice of certain alternative actions. I want to examine my construction of the interpretations I place on the students' behaviour and my own. My writing about these interpretations, which I have called 'stories', will then be exposed to other teachers and to other researchers.

A key component in my research has been the identification of appropriate tasks to offer to other teachers and researchers. These tasks have included mathematical problems, questions concerning transcripts and accounts of my

classroom experiences, and probes into the classroom experience of the respondent. I discuss these in greater detail in section 4, 'Communicating my Findings'.

Methodological considerations of other recent research studies in mathematics education

There are a number of research studies which, whilst being of interest to me from the point of view of their findings, have also been particularly relevant to my thinking about my research methods. They have helped me in addressing the question 'What constitutes convincing evidence in mathematics education? What are the sources of conviction?'

Use of Tasks for Readers

Schoenfeld and Arcavi (1988 op. cit.) write about the multiple meanings of the word 'variable'. Rather than try to explain each of the possible interpretations of the word and contexts for its use, they suggest activities which might be undertaken by individual readers, groups of teachers or students. These activities are an important part of the fulfilment of their intention to 'recapture some of the subtlety, and difficulty, of the idea of variable'. The word 'recapture' in this expression of their intention, confirms that they do not wish to share new knowledge with the reader. Rather they intend to refresh the reader's awareness of an important issue.

This is a short article written for a professional journal and does not claim to be a report of a research study. However it does illustrate an approach to a difficulty experienced by writers in attempting to convey an awareness or a sense, rather than a fact. Their approach is to suggest activities and tasks designed to offer the reader parallel experiences to those which the writers themselves have found instructive.

Argument Based on Individual Incidents

In her article entitled 'Difficulties with Mathematical Symbolism: Synonymy and Homonymy' Adda (1982 op. cit.) states

'Teaching and learning situations bring to light difficulties inherent to mathematics. Failures by students are signs of epistemological obstacles. So we are going to study our problem through paradigmatic cases, observed during mathematics classes' p205.

We are thus expected to acknowledge the incidents which she quotes as being 'paradigmatic', that is, as I interpret her meaning, as being representative of the general case rather than atypical. We can only be expected to do this if we have

sufficient experience of mathematics classrooms to judge whether these incidents are indeed representative. My feeling on reading this article was, again and again, one of recognition or resonance. I did not require any further evidence or argument that the 'failures by students' which she describes are typical.

Menghini (1994) makes similar use of her teaching experience to defend her argument about form in algebra. In suggesting that students typically do not make best use of form she quotes the results of a test administered to 16 year olds in Rome. In order to simplify

$$\frac{z^4}{c^2}(b-a) + \frac{z^4}{c^2}(a-b)$$

most students carried out detailed calculations, for example expanding brackets and collecting like terms. Hardly any students could answer the following question without carrying out some algebraic manipulation or comparing graphs.

'For which values of x is the following inequality true?

$$x^2 + x < x^2 + x + 1'$$

However, she merely *states* that 'Difficulties arise in analysis when $x+\Delta x$ is substituted for x , or in the proof of non-associativity of the arithmetic mean when it is required to find the arithmetical mean of $(a+b)/2$ and c '. The reader is expected to need no further evidence of this than their own experience. As a reader of this article I do not feel a different level of conviction about these statements on the grounds that one is backed by test result evidence and one only by assertion. Each is supported by my experience of students.

Arcavi again uses a similar approach in Arcavi (1994) where he lists eight behaviours which are characteristic of somebody with symbol sense, the analogue in algebra of 'number sense' in arithmetic. This list is compiled from observations of students by himself and his colleagues.

Again Sfard and Linchevski (1994 op. cit.) say 'Nevertheless, much evidence for the difficulty of reification may also be found in today's classroom In sections 3.2 and 3.3 we shall substantiate this claim with many examples' (p199). What follows in sections 3.2 and 3.3 is a large number of accounts of and interpretations of individual incidents.

Appeal to the Reader's Experience

At a conference session where Vinner presented his paper (Vinner 1994) I observed the reaction of other listeners to a similar process of drawing on

experience. The session began with the speaker relating some incidents from his own teaching which illustrated the 'syndromes' for whose existence he intended to argue. The audience found these accounts very funny. I am sure that at least part of the humour stemmed from the audience's overt recognition of the types of situations being described. In a similar way one technique used by stand-up comedians is to remark on situations which the audience will find familiar, in the hope that the audience has experienced but not explicitly noted the situation. Thus, one of the standard opening lines is 'Have you noticed how .. ?'. The research report, though, had a serious point to make, which carried conviction only if the reported incidents could be accepted as typical.

Each of these writers have used classroom incidents in a way similar to my own. That is, they used them to illustrate a point in the expectation that they will both be accepted as typical and as worthy of attention. It is not clear whether the deliberate collection of such incidents was part of the research method of each writer.

My intention in drawing your attention to the use made of individual instances by these several authors is to indicate that such use is widespread. I cannot and have not persuaded you that it is widespread merely by quoting a sufficiently large number of authors. I have (if at all) convinced you by reminding you of the many other research reports and articles you have read in which the authors have made similar use. Thus my method of argument is the same in this section as in the thesis as a whole. I do not seek to convince merely by argument but by appeal to experience.

Appeal to the Researcher's Experience

In chapter 2 I wrote about the work of Furinghetti and Paulo (1994), who undertook a survey of nearly two hundred students. The relevance of this work to me was that it drew my attention to some phenomena visible in students' work in second variable situations. In particular they state that

- students feel confident with questions requiring computational processes
- some letters elicit a stereotyped expectation of role and this causes students to lose their control of the semantic situation in concentrating on syntactic aspects

From the point of view of method, however, I contend that, even though they tested a large number of students, the authors find it possible to make such assertions on the basis of experience of working mathematically with students rather than on the basis of their test results alone. On p371 they say, of two test questions in particular,

'We considered it particularly significant to point out the high rate of confidence students show in answering these questions ... which we ascribe to the computational side of the questions proposed. The data we have on the other questions where focus shifts to more critical aspects show a greater level of doubt.'

Perhaps the students' greater confidence in giving answers which required computation was confirmation of a phenomenon already consciously known to the researchers. Or perhaps it served to highlight a more general feature of student confidence which was known to the researchers through their teaching experience but not previously explicitly acknowledged. It seems hard to believe that the phenomenon of student confidence in computational processes was unknown to the researchers before they undertook this research.

On p368 they state

'Since we have often observed that students, even if able to manage a satisfying "syntactic manipulation" of formulas containing parameters, do not grasp the "underlying semantics", in our research report we shall investigate this last point'.

On p373, speaking about students' stereotyped expectations of the roles of certain letters, they say

'these data confirm our initial observation about the two levels (the manipulative and the conceptual)'.

Here they are clearly using the evidence of the test results to confirm a phenomenon which they had already observed concerning their students.

My contention is that percentages of correct solutions in tests, and some students' brief comments on their solutions, serve to confirm or highlight the researcher's pre-existing awarenesses about her students' thinking.

Choice of Research Setting

One of the researcher's most important decisions is how to create the situation in which she can work on her awarenesses. Furinghetti and Paulo chose to set a test to students and examine their responses. A common decision is to design a teaching treatment and observe students undergoing this treatment (for example, Kieran 1994, Pozzi 1993b). Hewitt's decision (Hewitt 1994) was not to create any artificial situation but to use his teaching and everyday life to sharpen his awareness.

My decision was to place myself in classrooms with learners and teachers. I would observe the students and teachers at work together and take part in their

learning by intervening with individuals and by teaching the class myself. I would place myself in the position of learner by observing myself working on mathematical tasks. I would consider the contributions of other researchers in modelling the ontology and epistemology of the mathematical areas I had chosen. I would make opportunities to expose my data and constructions to other teachers and colleagues.

How Do I Learn in Mathematics Education?

Learning by Noticing and Marking

As I listened to the students I worked with I recorded incidents from the classroom which seemed interesting or typical. In order to explain what I mean by typical I will use the terms 'noticing' and 'marking' developed by Mason in Mason (1996). As a teacher in a classroom I am bombarded by far more sensory input than I can consciously respond to. Some of this input is remembered at a subconscious level only, that is it may later be recalled by some other stimulus but is not available for spontaneous recollection. These things are noticed but not marked. Other events are consciously marked and are available to be remarked upon. Part of the purpose behind the 'Discipline of Noticing' is to raise to the level of marking what has previously only been noticed. This is what I mean by an incident seeming typical. Similar particular events, or a generalised similar event, which have only been noticed are recalled to memory by the marking of the current incident. Mason (op. cit.) gives an example from a teacher's diary:

'In the midst of solving an equation in front of pupils:

Teacher: You divide by three

Pupil: Why do you divide by three?

Teacher: See, it works!

The pupil did not seem satisfied. Suddenly it came to me (I marked) that what the pupil wanted to know was "how do you know what to divide by?" and not "why do you divide by a number?". I knew in that moment that I must have misresponded to pupils countless times. I had noticed but not marked sufficiently to act or question further evident pupil dissatisfaction. I resolved to watch out for a reactive interpretation of pupil "why?" questions, and to work with them on framing more precise questions"

Most of the incidents which I recorded were striking because of a similar feeling of recognising an event which I had experienced many times before. A student's question, expression or reaction to a particular piece of mathematics

made me realise that I had seen or heard something similar on many previous occasions. This marking on the basis of having previously noticed but not marked also allows others to re-enter their own experience on reading my accounts.

Question - 'What is it like?'

My first question 'What are students' experiences of facing second variable situations?' could be phrased 'What is it like to be a student facing one of these situations?' How could I go about finding an answer to this question and what kind of answers could I give?

It is possible to argue from a relativist, subjectivist or radical constructivist point of view that it is not possible to know what somebody else is feeling.

'How can I know about another's mental states and processes ... ?' ... is very problematic from the perspective of a subjectivist epistemology (Ernest 1995). (p6)

A subjectivist position argues that we are each trapped within our own thoughts, understandings and feelings and have no means of real contact with the thoughts, understandings and feelings of others. But if we accept a wholesale subjectivist position then we may as well abandon the enterprise of teaching. Clearly it is possible, in some sense, for human beings to communicate. The mechanism by which we come to hold 'taken-as-shared' meanings is still mysterious. One thing that is clear is that it depends on our acting as though shared meaning can be achieved. This idea is explored by Sperber and Wilson (1986). They argue that one criterion for successful communication is 'relevance'. Relevance concerns the degree to which the information affects the context of the communication, tempered by the amount of processing energy required. Their main argument is that ostensive communication occurs when the communicator's intention to communicate is mutually known by communicator and addressee.

I choose to act as though I can achieve some understanding of what another is thinking and feeling. (This is not to say that I believe such an understanding can be fully achieved). Further, I choose to assume that I can suggest activities to colleagues which may assist them in moving closer to an appreciation of another's experiences.

I use a number of strategies for moving towards such an understanding. First I listen to what students say. This is so obvious as to seem facile and yet, in my sense, is a very complex matter.

Listening to students

In the first place, in a busy classroom a teacher's attention is engaged by many different concerns simultaneously. To hear what students are saying about the area of interest to me I need to be attuned to that area. I need to be able to attend to it as well as being concerned with social issues, who is following what is happening in the lesson and who is thinking of something else, who has done yesterday's homework, the absence of certain students, my own mathematical understanding of the problem in hand, whether we have enough graphic calculators between us to be able to use them on the next problem, why a particular student is so quiet today, and so on. It is a common experience for an observer in a classroom to mark events which the teacher has not seen, not noticed or not marked until their attention is drawn to it. In order to be able to hear what students are saying about their experience of second-variable situations I need to be attuned to listen to it within a complex collection of contextual factors.

In the second place a student's utterance gains significance, and thus is heard, against the background of the hearer's previous experience.

Several aspects of this experience are important. First, there is the hearer's experience of that student and that classroom. An utterance might be heard quite differently according to whether it is typical or atypical for that student or for that classroom. It might be recognised as using language which had become commonplace in that classroom or as being expressed in very unusual language. Interpretations of it might be made on the basis of the hearer's knowledge of the speaker's previous mathematical experience.

Next, the hearer's experience in other classrooms with other students working on similar subject matter allows her to hear particular utterances as representing the expressed experience of many students. It is this aspect which makes it possible for other teachers to recognise part of the significance of the utterance as perceived by the hearer.

Finally, the hearer's experience of and thinking about the mathematical area in which the student is working frame the way in which their utterance can be interpreted. The more different approaches to this area the hearer has experienced, the more possible interpretations will be open to her. Of course a student's utterance may also be the trigger for the hearer seeing the mathematical problem in a different way from any in which they have seen it thus far.

In the third place students' utterances have always to be interpreted keeping in mind the social and affective context. Leron (1995) claims:

'Traditional analyses of students' productions usually try to explain students' misconceptions as if it were merely (or mainly) a cognitive process, in which students are trying to solve problems by using as best they can logical thinking and their previous knowledge. Our study stems from the feeling ... that such conventional analyses of students' productions fall short of describing the student's mind in all its richness and complexity. The main problem is the strong emphasis on cognitive aspects, and the consequent neglect of affective and social factors in analysing students' productions' (p89)

Leron goes on to suggest that many of students' utterances in conversation with researchers can be interpreted as coping strategies rather than as attempts to engage with the mathematics of the problem. For example strategies such as holding on to something familiar and trying to say what the instructor expects are described as 'coping'. The researcher who knows the students as a teacher knows her students is more likely to be able to distinguish responses dictated mainly by coping strategies from those dictated by cognitive issues. She is more likely to be able to work with students in an environment where 'coping' can be set aside in favour of struggling with the mathematics of the problem.

These three aspects of the complexity of listening to students guided the way in which I chose to work with students. Because I wanted to know them as students and for them to behave as naturally as possible with me, I thought it was important for me to teach them and take on as many as possible of the roles of a teacher with respect to them. However I recognised that when I was teaching it was difficult for me to hear and observe as much as I might if I were not concentrating on running the lesson, so I also spent some time observing the classes with their usual teachers.

Working on my own mathematics

In order to move closer to an understanding of the learner's experience I undertook to put myself in the place of the learner in that I would be facing unknown situations in mathematics. I wanted to look for my own 'parallel experiences' to those that I witnessed my students having.

My understanding of 'parallel experience' is closely tied to that of 'resonance'. This term was used by Davis in her writing on the Discipline of Noticing, for example in Davis (1990)

'But how can you get someone else to have your experience? All you can do is look for resonance in their experience.'

It is also used frequently by Mason, for example here in a description of a personal experience

'What struck me later about the interaction was the not-hearing. I felt a resonance between this incident and many others I have participated in.'
(Mason 1994)

These two quotations illustrate two aspects of the use of the word 'resonance' within the Discipline of Noticing. The first describes a hoped-for response by a colleague to an exercise or account, whilst the second describes a personal response to a classroom event. 'Resonance' is a feeling of recognition and as such is not definable. It is often accompanied by a surge of energy or excitement. It might be identified by a verbal signal, 'Oh yes!', 'that reminds me of ..', 'it was the same when ..' or 'that happens to me all the time'.

Mason has borrowed and adapted the term 'resonance' from Skemp who introduced it in Skemp (1979). In speaking of how concepts are re-called from memory and activated Skemp uses an analogy of a net. If one node is lifted to the surface all the attached nodes are also brought nearer to the surface.

He invokes the idea of resonance to build a more complicated model of retrieval of stored schemas which accounts for recognition of examples and non-examples, among other things (p131). External stimulus 'vibrating at the same frequency' as 'the conceptual pattern stored within memory sets it vibrating. This is experienced as recognition.

My term 'parallel experience' describes an experience which gave rise to a feeling of resonance in me. In the middle of working on some mathematics I might recall a recent classroom incident where a student seemed to be expressing the same feelings that I was then experiencing. Or in listening to a tape of a student talking about his work I might recognise an expression of something I had felt myself when working on some mathematics.

I do not mean that my experience was the same as that of the student. In particular, my experience and the students' of the same piece of mathematics would be very different. What I am looking for in a parallel experience is a response in myself to a piece of mathematics at my own level which feels similar to the student's response to their task.

Pedagogical Analysis

At the outset of my study I had selected a particular mathematical area on which I wished to work. An analysis of that mathematical area was an important part of my study in several respects, amongst them that of understanding the experience of students.

First I wanted to identify frames through which students might view this area. How would they connect it with previous knowledge? What would be their interpretation of what this area of mathematics was about? An analysis of the mathematical area would give me a background against which to hear what they said.

Secondly I wanted to look at the language used to describe the concepts involved, both at the level of individual words and of phrases. How were these words used by professional mathematicians, by teachers and by students? Were there certain phrases whose use was fundamental to an understanding of the subject area?

Thirdly I wanted to study the historical roots of the subject matter. What were the shifts in thinking which brought about the modern view of the subject?

In my study of each of these aspects one of my aims was to give myself a variety of frameworks through which to view the students' mathematical work. I wanted to have a variety of ways of interpreting their understanding as evidenced in their work.

The Nature of Data

My data consists of transcripts and accounts.

The transcripts are of

- conversations between students

 - recorded in lessons where I was present in the room but not always with the tape recorder

- my conversations with students,

 - in the course of a lesson or informal advice session
 - in a specially arranged interview

- my conversations with teachers

 - individually with teachers of the classes with whom I worked
 - in groups formed specifically to look at my research

- my conversations with colleagues

 - at meetings in which I was invited to make some input concerning my research at conferences or other research *fora*

The accounts are of incidents

- from my classroom before I began my period of funded research

- from the classrooms in which I worked during the period of funded research

- from my undergraduate classroom after the main period of data gathering

- from my meetings with teachers

from conferences and other meetings with colleagues
from my own work on mathematical tasks

My explicit use of accounts of incidents as data is an important feature of my method and is worthy of some attention. How do I answer the concern that my accounts are 'merely' my judgement of events and not sufficiently objective? I will return to this concern explicitly at the end of this section, after a broader discussion.

Mason's account of the Discipline of Noticing (Mason 94 op. cit.) speaks of 'brief-but-vivid' accounts of incidents. The writing of these accounts allows 're-entering of salient moments from the recent past' (p12).

In Ernest (1995), Ernest queries whether it is possible to provide brief-but-vivid accounts of incidents.

'Whilst an individual may possibly construct expressions which symbolise and help to re-evoked a personal experience, the assumption implicated in claiming so cannot be easily validated. The memory against which the 're-evoked experience' is judged, the very means of judging validity, may be altered by the process of constructing a 'brief-but-vivid' account' (p5).

It is not clear to me that the validity of such an account depends on a comparison with my memory of the incident. As Ernest points out, memory can be a fickle thing. In the first place, our memories of an event are our memories of our experience of that event, so that judgements and emotions can get entangled with the words which were spoken and the actions which were performed. In the second place, our memories may later become memories of recalling the event, perhaps audibly in words or just internally. There are a number of incidents from my childhood which have become part of our family folklore, so that the original memory has been lost in the re-telling. Similarly, incidents from the early part of my research have faded in my memory, to be replaced by memories of describing the incident to other groups. So, at the moment of production, the brief-but-vivid account is a representation of my memory of my experience of an incident. The structure of my memory of that incident is now inevitably altered by the stressing of some aspects and neglecting of others in producing the account.

However, what has been stressed, and therefore retained, in the account is the aspect of this incident which initially caused it to attract my interest. The 'data' which survives the temporal passing of the incident is a collection of words which (in the ideal case) re-evokes for me that, and why, the incident was deemed worthy of recording.

Clandinin and Connelly (1994) express the concern that experience is used ubiquitously in talking about education but 'it is mostly used with no special meaning and functions as the ultimate explanatory context' p414. They go on to explain how criticisms of the notion of studying experience have led them to study it in the form of narrative.

They state further that

'when persons note something of their experience, either to themselves or to others, they do not do so by mere recording of experience over time, but in storied form Experience, in this view, is the stories people live. People live stories, and in the telling of them reaffirm them, modify them, and create new ones.' p415

In this sense my accounts and those of others, as well as my interpretations of these accounts, are stories which I tell and hear, and in the telling modify my overarching story of 'what it is like'.

The brief-but-vivid account is a form of representation of an incident and as such can be compared to other means of data collection, for example audio recordings of speech, video recordings of speech and action, written records of students' work and observers' field notes. Each of these means of recording captures some aspect of the incident and misses others. They differ over the agent of selection of those aspects to be recorded. Tape recordings can give the impression of being more objective, since the researcher has less control over what is captured and what is lost, but it remains the case that much is lost. It is tempting to think that in studying a tape recording we are studying the event itself. If we define an event as the combined memories of the participants, then a brief-but-vivid account is arguably closer to the 'real event' than a transcript.

Our response to 'repeatable' external stimulus can change with time. I sit in a meeting with a tape recorder running. At the end of the meeting I know that at least half a dozen points of real interest to me have been raised and that I will be able to work on them when I play back the tape. When I listen to the tape there is nothing of interest at all. Colleagues report having put down a very stimulating book or paper because they do not have time to work on it now. When they pick it up again, even if it is as early as the next day, they cannot remember or locate what was interesting about it. At a greater distance our general recollections of a series of events can become impressionistic. For example, I recall reading a transcript and being sure, having read it, that it recorded a number of instances where Trevor (a student) expressed an intention to, but did not, substitute a particular expression into an equation. When I re-read the transcript, in order to find the exact line references to which

to refer the reader, I found that none of the instances I had had in mind provided incontrovertible evidence that he had this intention. Yet my impression on reading the whole was that he had.

These are not just indications of a poor memory. They are indications that our judgements of spoken and written language depend on aspects of the situation in which we hear or read them as well as on the words themselves.

So the brief-but-vivid account as a product of data-gathering is not alone in its capacity to change in significance over time. In fact it may be less prone to change in its significance to me as time passes because the selection of what to record was made by the person who judged its significance, rather than by a feature of the technology used to produce it.

There are significant differences, then, between accounts and tape recordings as forms of data. The extent to which I control which aspects of an incident are recorded is much greater, especially at the time, in the case of accounts. In the case of tape recordings the decision to record must be made before the incident, and I have no choice as to which aspects of the incidents I record in this form. Facial expression, body posture, events elsewhere in the room are all unavailable to an audio tape recorder. I do have the power of selection after the incident, in that I can decide which parts of the conversation to transcribe and analyse, and how much detail to include in the transcript about voice tone, pauses and other non-verbal aspects.

Transcripts are, at best, a record of the words which were spoken. (And they fail on this count because the conversations included some words which were not audible and some sounds which were difficult to transcribe). They allow a reader to form a partial picture of what it was like to be in the situation, much as a classroom account does. My picture, as researcher, of the situation in which the recording was made is much fuller. On the few occasions when I left my tape recorder running on a student's desk rather than carrying it around with me I found it much more difficult to recognise any salient incidents in what was recorded. The reader's experience of my transcripts will have much more in common with my experience of these latter tapes than of those which recorded events at which I was present.

There is also a difference concerning the issue of authenticity. A sceptic who doubted that my transcribed conversations did in fact take place, or that I have faithfully transcribed them, could listen to my tapes. A sceptic who doubted that incidents I have recorded were genuine would have to seek out and consult the other participants in order to be convinced.

Finally I return to the question I posed on p65, that is, the lack of objectivity in my accounts of incidents. I do not seek to argue that my accounts will be objective. Rather I argue that other forms of representation of incidents are also subjective. I also argue that learning, including the production of new knowledge, takes place in capitalising on the subjectivity of the learner.

If it were possible to collect and record data in a completely objective way the results would be of little interest and no use in their raw form. Any new knowledge which is available from a study of this data is available only through largely subjective processes. These involve the bringing to bear of earlier knowledge and experiences to form explanations and judgements.

Subjective choices must be made concerning the collection of data and its interpretation. In choosing to record accounts of incidents as one form of data gathering I acknowledge myself as the research instrument and take advantage of my own subjective sensitivities in selecting what should become my data. This process is analogous to, though significantly different from, the process of selecting from a tape which parts to transcribe. Both processes involve being struck by the significance of an incident (in vivo or in listening to the tape) but they are very different. During the incident itself there are numerous other things which could claim my attention and distract me from the significance which I have glimpsed. In listening to a tape I am at leisure to hear some parts repeatedly and to weigh the significance to me of the words.

Being Convinced and Convincing

Conviction is a matter of empirical evidence, reason and fit with prior experience. The relative importance of each of these factors depends on the person who is to be convinced and the nature of conviction.

A brief case study will illustrate and amplify this point. In the sixties, seventies and early eighties in this country there was an enormous amount of political and intellectual interest in the relative merits of selective and comprehensive secondary schooling. Very large studies were carried out to compare the public examination results and other 'performance indicators' of pupils who had been educated in selective or comprehensive schools. (For example Marks (1983) and Steedman (1983)) Samples were carefully chosen and justified as representative. Statistical techniques were employed to remove the influence of other factors considered to be separate from the schooling effect, for example social class and previous academic attainment of the pupils. Nevertheless the studies came to different conclusions. Debate continued after the publications (for example in Marks (1984) and Woods (1984)) concerning the selection of the samples, the collection of data, the statistical methods used in its analysis and

the arguments used in reaching the conclusions. For those who wished to find fault with the conclusions of a study, it was a relatively easy task to find something to criticise in its design. The issue of validity became a matter of opinion, with much more resting on the reader's previous convictions than on the case made by the researchers. This debate was unusual in its extremely controversial and political nature. Nevertheless I think it illustrates the point that readers' conviction as to the validity of the results does not simply depend on the statistical case that is made. In the area of education research, as in many others, it is not possible to convince readers of the trustworthiness of one's conclusions purely by consistent reasoning.

My research seeks to capitalise on the reader's experience, rather than asking them to ignore it. I have not involved large numbers of students in my work, but have worked closely with a small number. My aim was to interpret their actions and work. I would work in situations which would be familiar to other teachers and which might be described as typical. In this way my interpretations might be seen by other teachers as transferable, that is as being relevant to their own situation, through speaking to their experience. My results are not any more generalisable than the reader admits them to be. If she chooses to say 'my students are not like that' then there is little I can do to persuade her otherwise.

Meanings of Experience

I refer a lot to my *experience* and that of the reader, of other teachers and of students. When I use this term I am not intending to refer to actual events, but rather to the individual's construal of events in time, at one of three levels. First, from inside the event I notice some things and do not notice others, I stress some things and ignore others, either consciously or unconsciously. Secondly, in standing back from the event I go further in processing my recollections, in categorising, reconstruing and laying this particular memory alongside other individual events. Thirdly, if this event becomes for me, by conscious or unintentional process, an example of a particular type of incident, then it forms part of a bed of experience which is one of the sources of my being as a teacher.

Moves from one layer of experience to another result from a drawing back from the immediacy of the event, from seeing it as 'what is happening around me now', to seeing it *as an event*, to seeing the event as one of a type. These stages of drawing back may take place over a very short or a very long period of time.

None of this ignores the possibility that my memory of a particular event changes over time. Every time I recall my memory of the event I re-process and potentially reconceptualise it.

When I read somebody else's research results the first thing I do is to check them out against my own experience. It is common for a group of teachers or teacher educators, in discussing a piece of writing together to compare anecdotes which seem to deny or confirm what the author is saying. This is true even if the members of the group have acknowledged the failure of anecdotes to prove general results.

In my writing I deliberately appeal to this reaction. I am asking my reader not only to decide whether I have argued cogently but to check whether what I am reporting fits with their experience of similar situations.

As part of my research, and through this thesis, I attempt to communicate with other teachers by contacting their classroom experience and their mathematical experience. I offer them accounts and transcripts of classroom events and of students working on problems. I offer them mathematical problems which have allowed me to experience confusion over constants and variables.

Listening to the Accounts of Others and Comparing Experiences

I have argued that when we compare our experiences with others we are seeking a comparison, not of events, but of our recollections of our experiences. Teachers' accounts which I have used in my work need to be seen in this light.

In asking teachers to give accounts of events in their classroom I am not calling on their recall of definitive versions of events held in their memory. Firstly I am asking for response to a prompt, a request in some form for a type of event, a parallel event, an exemplary event. So the request forms a framework for the recall of the event, and the recalled event is shaped by the request. Secondly, the request is made in a social situation. The respondent is conscious of her social needs in her response. These may include a need to give evidence to the group about the kind of teacher she is - to create an identity for herself within the group. They may include needs concerning her acceptance within the group - a need to conform to what she sees as the group norms. They may include desire to express a moral viewpoint or a position on the rights of pupils, for example. They may include need to claim entitlement to status, preferential treatment, particular attention etc. For some individuals these needs may exist in respect of the discussion leader, or some other individual within the group, rather than the group as a whole.

Middleton and Buchanan (1994) explore some similar issues in their paper which suggests that reminiscence is not merely a matter of remembering. They claim that the act of remembering as a communicative action can be analysed in terms of socially accomplished activities. They identify the domains of effect of

conversational remembering as 'Situated identity', 'Group membership', 'Cultural and moral orders' and 'Entitlement claims'. An example is given of a transcript of a conversation between an elderly lady and her peers, in which she justifies her attitude to drinking by giving an account of an event from her youth.

In listening to teachers I need to practise interpreting their accounts in the light of the influences which may have produced them. In particular, I am aware that the teachers in the two groups with whom I met may have wished to give the impression that the sessions they attended had a significant effect on their thinking. I have therefore treated such statements with caution.

Communicating my Findings

The findings or conclusions of my research are not 'facts' in the sense of scientific knowledge. They are not statements in defined terms which have validity in a determinable domain.

Rather they are 'slogans' in the terms of Komisar (1961). A 'slogan', they say, is a generalisation in the sense that it summarises proposals or exhortations, but not in the sense that these particulars can be deduced from the general. The detail which is summarised by the slogan must be given by the author, or its meaning is arbitrary. In other words they are forms of words which serve to summarise (hold together, recall, trigger) a larger corpus of knowledge and experience. They serve the purpose of allowing communication between people with some common experience.

Elliott expresses a similar sentiment concerning the use of exhortations to improve practice:

'General rules are guides to reflection distilled from experience and not substitutes for it' p50 (Elliott 1991)

Schön (1983) points out the distinction between theories which predict or explain and metaphors from which professionals may construct their own accounts of unique or changing situations. The latter are the results of what he calls 'action science' which aims at development of themes rather than theories. Torbert (1976) similarly speaks of action science in the following way:

'This kind of personal research can lead to an action science in contrast to a reflective science - a science useful to the actor at the moment of action rather than to a disembodied thinker at the moment of reflection.' (p167)

I aimed not to prescribe teacher actions but to suggest alternatives. I found that I was able to identify some of my habitual responses as a teacher and to look at

them freshly. I identified and used some alternatives. I aimed to enable other teachers to do the same.

Communicating my awarenesses

I have described how I went about increasing my awareness of the students' experiences in facing second variable situations. The means by which I have gone about communicating those awarenesses, and intend to communicate them through this thesis, is another aspect of my method.

The words I write in this thesis cannot carry meaning. It is not possible to convey my thoughts to you in my writing, despite the prevalence of the conduit metaphor for written communication. I do not subscribe to the view that the meaning I wish to communicate exists independently of the language I use to formulate it for myself. My meaning is shaped by language but not contained by it.

Nevertheless words are the main vehicle of this communication. If words do not literally convey meaning then how can we hope to communicate? The key process in communication is the summoning up of images. If communication is successful then the images summoned up by the words for the reader are those which the writer intended. Of course the images summoned depend on the reader as well as on the words. An account of a classroom incident will summon up quite different images for an experienced teacher and a non-teacher. Their responses to their reading will be influenced by their experiences. In addition the reader may have some control over the images they choose to entertain as a result of reading certain words.

Communication through this thesis

The main thrust of my communication through this thesis is deliberate appeal to the reader's experience and volition. Since it is not possible to communicate ideas through words, this appeal is made through tasks, accounts and transcripts.

The tasks I suggest are mainly mathematical. They are tasks which I have worked on and which I have included because:

- they illustrate a distinction which led me to a new awareness, or
- I felt that my experience in working on them was parallel to a student's experience, or
- different approaches to the task represent different views of the mathematical area which I want to highlight.

I do not anticipate that your experience of working on these tasks will be the same as mine or the same as those of other readers. The awarenesses which are heightened by working on the task will depend on the sensitivities of the reader. In particular I anticipate that you will connect your experiences of working on these tasks with your previous experience in ways beyond those which I had in mind.

However, in part the value of this thesis is measurable by the extent to which readers find that they are able to use the tasks I suggest to increase their awarenesses.

The accounts included in the thesis are mainly of events in the classrooms in which I worked as a teacher. A few are of my own mathematical work.

In each case what I am seeking from the reader is resonance or recognition. It is recognition of a commonality that allows an observation to be made. If we hear somebody else describe an experience similar to our own then we have the opportunity to become more aware of what our own experience is and what are some of the common patterns within it.

The role of the transcripts within this thesis is more problematic. They are not produced as evidence that, in general, students think, speak or understand in a certain way. Indeed the extent of the generalizability of any statement within this thesis is a matter for the reader.

They invite recognition and resonance in much the same way that accounts do. They also, rather more forcefully than do accounts, invite sense-making in the form of interpretation. They invite the reader to create stories and explanations for the words and actions of the participants in the conversation.

In addition, transcripts, much more than accounts, invite resonance with the teacher's feelings and motives. This assertion is based on my own experience of this type of resonance. I read an excerpt from a transcribed conversation in Schoenfeld et al (1993). The conversation was between researcher (JS) and interviewee (IN) and concerned their work on straight line graphs on a graph plotting package. Prompted to think of straight line graphs, IN has written down $y = mx + b$. When asked which letters she needed to put in values for, she said x and y , so that to start they got $2 = m3 + b$. Next she agrees that she needs values for m and b so that eventually they get $2 = 4x + 1$.

- JS Stuck?
- IN Umm ... I don't know why I'm having problems. I know how to graph, um ... OK, if we have, 'cause what I was trying to do ... you see, because lately I've been getting confused, this is the way I was taught to do it, put in, plug in things for x and find y and lately they've been telling me that I have to do some other way, like when you find the slope and then you do some counting process, so I kind of forgot.
- JS Use the fact of knowing the slope rather than?
- IN Yeah
- JS Rather than,
- IN The formula. But I forgot how to use this formula, but I feel more comfortable with it. Should I plug it into which one, into this [can't see what she points to]?
- JS OK, why don't we write the equation for the line that we had. Remember we're a little bit confused because this, we have a y -value of 2 already here. So the equation of the line would be $y = 4x + 1$, would be the standard form.
- IN Yes.
- JS OK. So then if you plugged that in, why don't you write that down. That might help. $y = 4x + 1$.
- [IN writes $y = 4x + 1$. Both laugh.]

I saw in this teacher/researcher's transcribed conversation a drive to get IN to write down the equation $y = 4x + 1$. I recognised a drive which I had felt to get a student to write down something which I thought should be an aid to their thinking. Reading the student's exact words allowed me to compare my own reactions as potential teacher with those of the actual participant. In this case, the feeling that I would have done something very similar allowed me to focus on what that action entailed.

Reading the student's exact words in the transcript allowed me to recognise the experience of the teacher in a way which an account might not be able to do.

My interpretations of the accounts, tasks and transcripts enabled me to formulate alternative actions which a teacher might take in response to noticing the types of events I have described. I anticipate that you, as reader,

will judge my interpretations and suggested actions by standards of cogency and in the light of your own experience.

Conclusion

In this chapter I began by comparing mathematics education as a field of enquiry with other fields. I drew parallels with literary interpretation and with professional knowledge which suggested ways of thinking of research in this field.

Next I considered several established paradigms and found that each fell short of being an appropriate framework for the kind of enquiry I wished to pursue. I described how I wished to make deliberate use of my experience as a teacher and to use my research as a means of informing my practice by offering alternative ways of acting.

In a later section I considered how other researchers and writers in the field of mathematics education had found ways of capitalising on subjectivity rather than aiming for objectivity. I used these examples to back my assertion that appeal to a reader's experience is a common and useful technique.

Finally, I described the means by which I intended to learn about students' experiences of second variables, discussed the nature of conviction, both my own and that of the reader, and made some remarks about the way in which this thesis might fulfil my intention to communicate what I have learnt.

Chapter four goes into more detail concerning how I set about the learning process.

Chapter 4 Description of Method

Introduction

What I describe in this chapter does not constitute an experiment designed to test a hypothesis. Rather it catalogues the ways in which I attempted to engage in mathematical thinking and activity and to expose myself to the mathematical thinking and activity of others in the areas I have described in chapter one. My aim was to place myself in situations where I might act as a practitioner in classroom teaching and as a learner of mathematics. I wanted to use as triggers for my own noticing the kind of situations which would seem familiar to other teachers. My accounts of these situations could then be useful to others.

My intention was to put myself in situations where I would be able to observe the work of and work with both students and teachers as well as looking at my own mathematical thinking. In particular I was concerned with work on the aspects of the first year 'A' level syllabus which I have already outlined. It was important that I become part of the students' normal learning experience . I wanted students and teachers to react seriously to what I asked them to think about and do, not to treat it as a game or as an activity which was only for my benefit.

The School Students

At the end of the academic year 92/93 I approached three schools in the first instance by writing to their head teachers and heads of mathematics. I asked for permission to work with one of their lower sixth classes and its teacher for the whole of the academic year 93/94. I gave a brief indication of my area of interest

and said that I would want to observe and sometimes to teach the class by agreement with their usual teacher.

One school turned the request down on receipt of my letter. The other two each invited me in to talk to a meeting of some members of the department so that individual teachers would have a chance to hear what I had to say. In each case one member of the department who was to be teaching an 'A' level class in the next academic year said that they would be willing to have me as a co-teacher in their class.

School A

School A is a very large, relatively new, comprehensive school located in a city. Behaviour codes are informal. For example, students address teachers by their first names and there is no school uniform. There are no bells for lesson changes or for beginnings and ends of breaks. The school prides itself on its care for the individual and on its curriculum innovation. Exam results are below national average but compare favourably with other comprehensive schools in the city. A number of the other city schools cater only for the 11-16 age range, so a proportion of the sixth form at school A have transferred from other schools. Mathematics is taught in mixed ability classes from age eleven to sixteen and students use materials from the SMILE scheme. In the sixth form all 'A' level students study four subjects in the first year and most reduce to three for the second year.

At school A the teacher I worked with was Peter. He was in his mid-twenties and held a post of responsibility for mathematics within the school. He was active in a mathematics teaching association and a professional association and was respected as a person with high standards, integrity and warmth. He had taught the same 'A' level course the previous year.

The class with whom I worked were studying for an 'A' level in Pure Mathematics and Statistics and Peter was teaching them the pure maths element of their course. There were fourteen students in the class.

School B

School B is a smaller school with a longer history and is situated in a small town. Its pupils are drawn from the surrounding rural area and the outskirts of a nearby city as well as the town itself. Behaviour codes are more formal than in school A. Staff are addressed as Miss or Sir, uniform is worn by students below sixth form and bells mark the changeovers between lessons. Exam results are below national average. They are higher than in comprehensive schools in the nearby city, but lower than those in the neighbouring small

towns. The maths department gives an air of efficiency and good organisation as well as stressing the importance of the individual.

At school B I worked with Nigel who was in his early twenties and in his second year of teaching. He had not taught any sixth form classes before. He was regarded as likeable and competent by the department.

The class with which we worked were studying for an AS level in mathematics. Their syllabus included some pure maths, statistics and mechanics. Nigel was to teach them the whole course over two years. There were thirteen students in the class.

Working Practices

Appendix B, 'Early Impressions', is my account, written at the time, of my introduction to and first experiences at the two schools.

In both schools I was with the class almost every time they met with their teacher. This happened twice a week in each school and the lessons lasted a total of three hours a week in school A and two hours twenty minutes each week in school B. Sometimes the class teacher led the lesson while I observed and spoke to individuals from time to time. Sometimes I led the lesson. Occasionally I was alone with the class when their teacher was absent on other business.

Whenever I led the lesson I agreed with their teacher beforehand what I would do. I also set and marked homework and at school A I consulted with Peter over the students' report grades. I tried to involve myself as much as possible with the life of the school as it affected these classes. I was pleased to be invited to contribute to an extra-curricular special event at school A and an open evening at school B.

I occasionally used a tape recorder in lessons which I was leading but more often I made notes in a notebook. I would write down what struck me during and after the lessons. If I made a tape I would listen to it soon after the lesson and make notes on or transcribe any incidents which struck me at the time of the lesson or in listening to the tape. There was time to speak to each teacher briefly once a week after the lesson and we made time for longer discussions every few weeks. Again I occasionally made a tape of these conversations, but usually I made notes immediately after them.

I was concerned that teachers and students would perceive me as working with them rather than on them. One way in which I pursued this end was to spend time explaining why I was interested in working with them. I also made a

policy that where there was conflict my role as teacher took precedence over my role as researcher.

It was rare for such conflict to arise and when it did it usually occurred as I tried to decide whether to intervene in a student's working or conversation. The teacher in me wanted to say something which might help them to switch to a more useful course of action while the researcher in me wanted to see where the erroneous line of argument would lead.

I hoped that both teachers and students would see benefit to them in having me in the class. For the teachers, although their preparation and marking time was reduced, their class contact time was not reduced and the time they spent in discussion with me balanced the time I saved them. Any benefit to them, then, was in having a colleague to work alongside them. For the students there was the benefit of having two teachers on call to help them both during and after lessons.

The two classes responded quite differently to my presence and teaching and by the end of one term I had decided not to continue working at school B.

Appendix C, 'Leaving school B', is what I wrote about this at the time of the decision.

At the end of the year I asked all the students in the class at school A if they would allow me to set them some problems and tape record what they said in trying to answer them. All agreed to do this but in the event two were unable to participate. Each interview lasted approximately one hour and the students worked on two or three problems chosen from a batch of six. I had selected the six problems from amongst those that I had used with individuals or other groups during the year. They were problems which had already provoked interesting responses from some students. I also conducted interviews with two boys from the same year group who were not in this class. These were boys whom I had taught in an extra-curricular setting and who were considered by their schools to be very able in mathematics.

The reasons for these interviews were:

- to return, with particular individual students, to a type of question which they had made an interesting response to earlier in the year, so that I could record and consider at length what they now had to say
- to get responses from different individuals to a problem which had elicited an interesting response from one member of the class or from some other student or colleague

- in the case of the boys from outside the class, to expose myself to the thinking of two students identified as very able.

Each of these interviews was transcribed and the complete transcripts appear as Appendix D. The transcripts which appear in the main part of the thesis are extracts from these complete records of the conversations.

The Teachers' Groups

At the end of the academic year 93/94 I began inviting local teachers to join me in a series of meetings which would start in September. I contacted local schools and the local advisor for names of people who might be interested in taking part and I wrote to them individually. The letter which I sent appears as Appendix E, 'Invitation Letter'. Six teachers from two different schools responded and we met three times during the following term.

During that term I set about forming another group in a different locality to meet during the Spring term of 1995. To this end I blanket-mailed all secondary schools with a sixth form in the area and received responses from eight teachers in four different schools. This group met five times during the Spring term and early Summer term.

At the meetings we followed three strands of activity. The first was mathematical activity. I suggested a number of mathematical problems to the group to work on and we discussed the mathematical awarenesses highlighted by these problems. The second was consideration of transcripts and accounts of incidents which had taken place at schools A and B the previous year. I would invite recognition and interpretations, in the senses which I described in chapter three, of these incidents. The third strand was discussion of classroom gambits. I would suggest a technique which the teachers might adapt to their situation and in subsequent meetings teachers would report on their experiences.

I tape recorded and transcribed most of the meetings.

The two groups were of quite different composition. In the first group most of the teachers were relatively inexperienced and only one of the group held a post of responsibility within the department. Four of the six were teaching 'A' level for the first time that year and one would not be teaching 'A' level until the following year. The meetings tended to be dominated by the two most experienced teachers. The fourth meeting planned in the series was postponed and then cancelled because of illness.

In the second group all the teachers had at least four years' teaching experience, although one was teaching 'A' level for the first time that year. Amongst the

eight were three heads of department, two seconds in department and one other with a responsibility allowance. The meetings were not dominated by any particular individual and all members made significant contributions.

Both series of meetings were characterised by friendly and relaxed debate and positive ethos. To illustrate this I quote from one of the group during the last meeting of the second series:

'I feel that having come - it makes one sit down and have time to think whereas without coming and meeting with a little group, you know, you just don't get round to it, don't have time to stop and think at all. I think the fact that we have gathered as a group has been positive.'

Other Groups

Colleagues

As well as working with these two classes of students and these two groups of teachers I had the opportunity during the course of 93-95 to invite several other groups of adults to work on my ideas, and in particular to work on mathematical tasks.

These groups included my colleagues at the Centre for Mathematics Education and a group of participants invited to a day conference organised by the Centre. This conference was one in a series held for the benefit of those pursuing a higher degree by research through the Centre. I also had the opportunity to work with participants at conferences organised by SNM (Stowarzyszenia Nauczycieli Matematyki, a mathematics teachers' association in Poland), by CIEAEM (Commission internationale pour l'étude et l'amélioration de l'enseignement des mathématiques) and by BSRLM (British Society for Research into Learning Mathematics).

In each case I asked the colleagues at the meeting to engage with a mathematical task or in reflection on a transcript or account. Subsequently I asked for their responses to particular aspects of the stimulus. I also invited responses to the method of working.

In some senses there is nothing unusual in a researcher asking colleagues to respond to and comment on their work at conferences. It is more unusual to incorporate it as part of the research itself and to explicitly acknowledge and make reference to the responses and comments that were made. The method that I have chosen to adopt in my research deliberately *includes* such feedback and acknowledgement as I have explained more fully in chapter three. For examples of such references see my comments on problem B (p91), on problem G (p111) and on problem N (p145).

Students

I also took opportunities to work with students at Open University summer schools, at three sixth form mathematics days and, on four occasions, at classes for very able thirteen year olds which formed part of a Royal Institution Masterclass series.

I was not able to choose subject matter for these classes specifically with my research in mind, but I was able to be sensitive to my research interests in listening to and interacting with students as they worked.

For examples of the outcomes for my research of such events see the discussion of expressing generality (p118), of problem K (p136), of unhelpful stereotypes (p154) and of problem S (p216).

Working on Mathematical Problems

A very important part of my method was to look in my own experience of working on mathematical tasks for parallels to what I witnessed students experiencing. I have explained what I mean by 'parallels' in chapter three.

Part of my research, then, was to look for and explore tasks which gave me this feeling of parallel experience and which might do the same for others. I found these tasks in conversation with colleagues, at conference sessions, by amending and extending tasks which I had used with students, and by asking myself questions about mathematical situations.

My notes about my experiences in working on these tasks were nearly always made after the event. The exception is my work on envelopes (see Appendix F, 'Envelopes'), where I made the notes as I worked. Sometimes my work with the students preceded my work on the mathematics, and I recognised the parallel as I worked on the mathematical task. Sometimes my work on the task preceded the work of the student, and I recognised the parallel when I later saw the students' work. On other occasions my recognition of the parallel was subsequent to both events.

Selection of Data for Analysis

My analysis of the transcripts and incidents I chose to work on is contained in chapters five to nine. These chapters contain no real indication of how these items were selected from the mass of data which was potentially available. My criteria for selection were almost entirely subjective. I recorded and/or analysed an incident if it struck me as relevant to my interest. My interest, of course, developed as my research progressed so that the choice of data in the later stages depended on the analysis made in earlier stages. Under these circumstances it is difficult to distinguish between collection, selection and analysis of data.

In the situations in which I worked I might have chosen to record many different things and a different person would certainly have recorded differently. It is not possible to present all of my work in such a way that an independent observer could decide whether they would have selected and analysed my data in the same way. I have, however, included, as Appendix D, the complete transcripts of my conversations with individual students from school A, from which I have quoted quite extensively in chapters five to eight. My reason for doing this is so that the reader can achieve a better informed picture of the personalities of the students and the nature of the conversations. I expect such impressions to affect the reader's interpretation of the incidents I present.

I make no claim that mine are the only or the best possible stories and interpretations. The test of the validity of these interpretations is their capacity to inform future actions.

In the following three chapters tasks, accounts, transcripts, interpretations and references to other writing are interwoven to produce the themes, metaphors and labels which constitute my findings. My 'data' is organised according to these themes rather than chronologically and, as a result, tasks, students and, occasionally, transcripts appear on several occasions through these three chapters. I have included in Appendix G, 'Tasks chart', a table which shows where references to each task and each student occur.

In the transcripts which occur throughout these chapters I have indicated pauses in speech by series of full stops. Each full stop represents a pause of approximately half a second. Grammatical pauses of less than half a second are indicated by a dash. In order to make the transcripts easier to read I have edited some of them by taking out responses from the listener ('right', 'okay' etc.) and small hesitations and repetitions. The original transcripts, which are given in Appendix D, report all the words which were audible on the tape.

Chapter 5 Particular and General

In this chapter I look at some ways in which acts of generalisation have been categorised and examine the relationships between the 'types' I have identified. I go on to examine further the role of examples in acts of generalisation and, especially, the notion of the 'generic example'. Finally I discuss expressions of generality by learners and the understanding by learners of general statements made by others.

Types of Generalisation

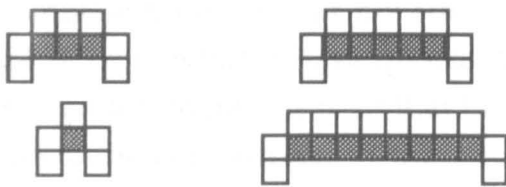
In chapter 2 I discussed contrasts between the traditional idea of concept formation by abstraction from a large number of examples and ideas on the role of generic examples or paradigms. In particular I explored some writings on the subject of paradigms (Freudenthal 1978), prototypes (Lakoff 1987), 'on-the-spot' generalisations (Krutetskii 1976) and generic examples (Mason and Pimm 1984, MacHale 1980, Hazzan 1994, Harel and Tall 1991).

In this chapter I want to draw attention first to different ways of arriving at general statements concerning problem situations.

Two Examples

Discovering Rules

Consider the following extract from a pupil's textbook (SMP 1983).



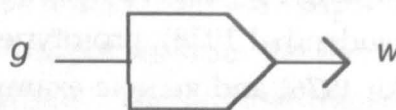
g	w
3	7
5	

(a) Copy this table and fill it in for the patterns above. g stands for the number of grey tiles. w stands for the number of white tiles.

Draw three more patterns of the same kind. Fill in the table for your patterns.

(b) If there were 100 grey tiles, how many white tiles would there be?

(c) Draw a machine chain for the rule.



(d) If there were 400 white tiles, how many grey tiles would there be?

In working on this problem from an SMP 11-16 textbook a student is invited to consider several special cases and to make a generalisation about the relationship between the number of white tiles and the number of grey tiles.

A student who followed the directions in the text might complete the table and notice that the number of white tiles is always four greater than the number of grey tiles in the table. Her attention is on the numerical results of her empirical investigation, that is the numbers in her table and the relationships between

them and any generalisation she makes is based on empirical evidence. In seven particular cases the number of white tiles is four greater than the number of grey tiles, so it seems reasonable to assume that this will be the case for any number of grey tiles.

By contrast a student might arrive at such a conclusion without the need to produce any table of results. Looking at the diagrams given in the book, she might see the 'sameness' in the patterns in such a way that the relationship between the numbers of grey and white tiles becomes visible. One such 'way of seeing' is

- make a row of grey tiles
- add a row the same length of white tiles above them
- add four more white tiles, two at each end.

From this way of seeing the construction of the diagram it is a small step to see that the number of white tiles will always be four more than the number of grey tiles.

This example serves to illustrate two different routes to the generalisation, one made here by looking at the relationships between the numbers in the table and one made by 'seeing' the way in which the diagrams are constructed.

How Many Triangles?

A further example, at a different level of mathematical difficulty, will expand on these differences.

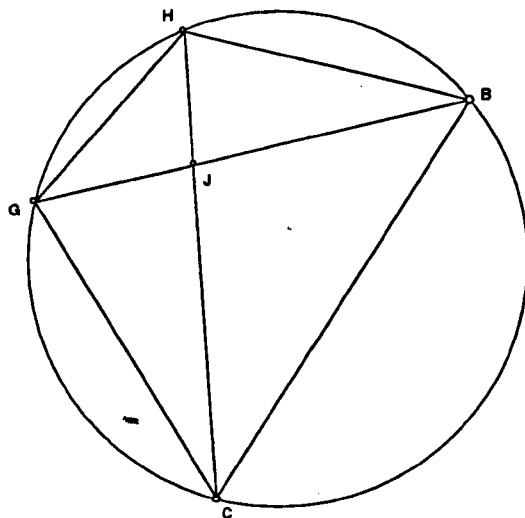
During the 1994 SNM (Stowarzyszenia Nauczycieli Matematyki, a mathematics teachers' association in Poland) conference in Poznan, a group of us worked on the following problem:

Problem B Suppose that n (≥ 4) points are chosen on a circle, and each pair of points is joined by a straight line. Assume that no three lines meet at a point except on the circle. Let p_n be the total number of triangles formed within the circle. Find a formula for p_n for $n > 6$

I have also asked colleagues to work on this problem on two other occasions.

Below is an account of my work on the problem and some reflections on the process I went through. You may like to work on the problem yourself before reading any further.

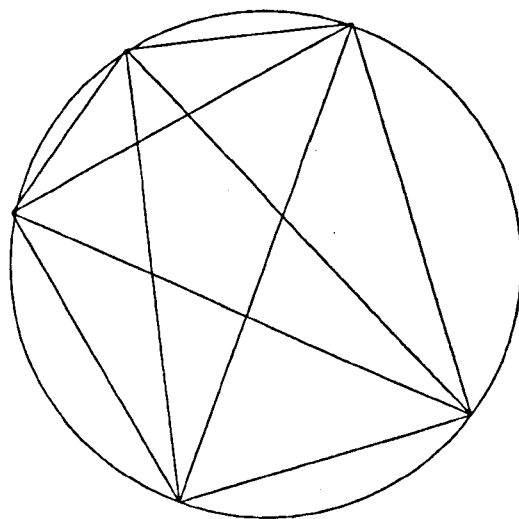
I started with this diagram for $n = 4$.



and counted the eight triangles HBJ HGJ GJC CJB

HCB HCG GCB GHB

I thought it might be useful to categorise them as having all three or only two vertices on the circle. I moved on to $n = 5$.



I felt I needed to be more careful in counting these so I deliberately chose to count first those with three vertices on the circle. I found ten of these. As I was counting them I realised that I was doing the equivalent of choosing three points from five for each triangle. Whilst thinking about this *particular* case I could see that in the *general* case the number of triangles with all three vertices on the circle would be $\binom{n}{3}$.

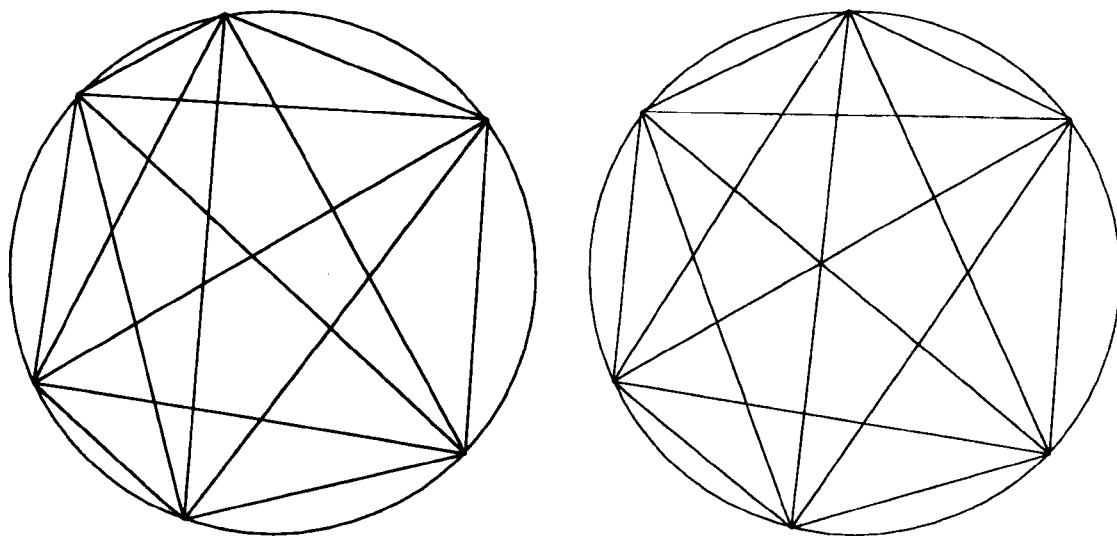
Next I thought about those with two vertices on the circle. As I looked at these I realised that each of them had one vertex at one of the five points of

intersection within the circle. Each of these five points of intersection was the vertex of four different triangles whose other two vertices were on the circle.

So I counted twenty such triangles, but at this stage I failed to see the counting in a way which would enable me to count in the cases $n = 6$ etc.

Finally I counted triangles with only one vertex on the circle. I found that there was only one such triangle corresponding to each point on the circle - five in all. Again I did not think ahead to the general case. It seemed too hard at that stage.

So I had a total of thirty five triangles for $n = 5$ and I moved on to $n = 6$.



Here I understood the reason for the condition, 'Assume that no three lines meet at a point except on the circle'. In my first diagram the condition is fulfilled, but in the second diagram it is not. The triangle with no vertices on the circle is absent from the second diagram because the three lines which had formed it now meet at a point.

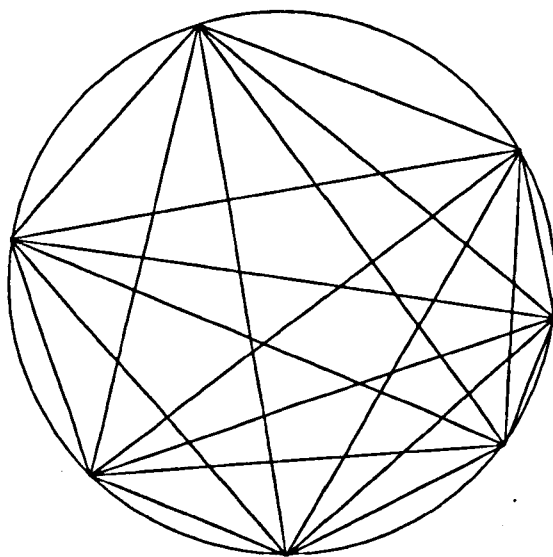
Using the first diagram I began to count. I could easily count my first category of triangles - those with three vertices on the circle - because I had worked out a general method already. It had to be $\binom{6}{3} = 20$.

When I began to look at those with two vertices on the circle I found that my earlier method was only partly applicable. It was again true that every point of intersection within the circle was a vertex of four triangles whose other two vertices were on the circle. But how many of these points of intersection were there? I now had three particular cases to look at in formulating my general rule. Eventually I realised that every point of intersection required two lines which did not have an end point in common, in other words that each point of intersection was defined uniquely by selecting four of the original points on the

circle. So the number of points of intersection was $\binom{n}{4}$ and the number of triangles with two vertices on the circle was $4\binom{n}{4}$.

Next I moved on to triangles with only one vertex on the circle and again I looked at each point on the circle to decide how many triangles had that point as their only vertex on the circle. I did not find a general method but decided that each point had only two such triangles so that the total of such triangles was twelve (I later found that this was not true). Finally there was one triangle with no vertices on the circle. This made a total of 93 triangles for $n = 6$.

By this time I thought I had a general method almost completely worked out but I decided to look at $n = 7$ for some kind of confirmation.



I satisfied myself that I needed $\binom{7}{3}$ and $4\binom{7}{4}$ for the numbers of triangles with three and two vertices on the circle, respectively, and thought that I would want 7 times 3 triangles with one vertex on the circle. (This was because I thought there were 3 such triangles at each original point). However when I looked at the diagram for confirmation I found that there were far more than three such triangles at each point. So I had to think again about how to count them. *How* I managed to do this is now completely lost to me but I have reconstructed some reasoning as follows by looking at my answer. Make a selection of five points on the circle. One of these will be a vertex of the triangle. Call this point A. Another two will be the far ends of the two lines from A which form two sides of the triangle. The last two will be the two ends of the line which forms the third side of the triangle. I had to convince myself that for any group of five points this can be done in only five ways, i.e. that once I had chosen which

point will be a vertex of the triangle there was only one triangle that could result. The number of triangles with one vertex on the circle is then $5\binom{n}{5}$.

I was then well set up to notice that in general the triangles with no vertices on the circle were defined by a choice of six points. I spent some time playing with a diagram to see why a selection of six points gives rise to only one triangle.

So finally my formula for the total number of triangles in the case of n points is $\binom{n}{3} + 4\binom{n}{4} + 5\binom{n}{5} + \binom{n}{6}$.

Although I have included in my account some indication of the mistakes I made in solving this problem, it gives no real impression of the complexity of the process. It does, however indicate the main stages of my working.

My reason for being interested in this problem is to do with the shifting of attention from particular to general and back which seems to me to be essential to its solution. All of the colleagues whom I have invited to work on this problem have found it necessary to draw a diagram for a particular value of n in order to make a start on this problem. I have found very few who do not work out the value of p_n for at least one value of n . However a purely numerical approach to this problem is unlikely to lead to a solution. It is by no means certain that accurate results can be obtained empirically - as soon as n becomes reasonably large it becomes difficult to arrange the points around the circumference of a circle in such a way that no three lines joining them intersect at a point. Even if this is achieved, counting the triangles accurately requires some strategy, that is some thought about the order in which the triangles are to be counted.

If all of these difficulties are overcome this table of results might be obtained:

Number of points	Number of triangles
3	1
4	8
5	35
6	111
7	287

In order to obtain any generalisation by using familiar pattern-spotting or 'difference' techniques I would need a great many more results than this. The formula I derived above is a polynomial of degree six, so that the set of results

above is not sufficient to reach a constant difference by repeatedly taking differences. Neither is it sufficient for establishing what the polynomial is, even given that it is of degree six.

And yet the question in its original form invites solvers to find p_4 , p_5 , and p_6 before attempting p_n . The purpose of looking at these particular cases is not simply to generate data but also to examine the general structure of the relationship between the number of points and the number of triangles. I could achieve this examination by attending to the way in which I count the triangles in the particular cases. A generalisation made in this way might be described as 'seeing the general in the particular'. Recognising the need for, and establishing, a way of categorising and then counting the triangles are essential stages in solving this problem. It is necessary to *look* at the particular but *see* the general. The solver must work with a diagram for, for example, $n = 6$ but at each stage separate what is true because $n = 6$ from what would be true for any value of n .

These two examples, 'Discovering Rules' and 'How Many Triangles?', illustrate two ways of using the particular to move towards the general. One way uses the results of specialising as empirical data and attempts to generalise inductively. The other uses the particular as generic and attempts to see the general in a single particular case.

Seeing the General in the Particular

Mason, Burton and Stacey encouraged this kind of use of specialising in Mason et al (1985). Speaking of the solutions to two particular problems they say

'Now, perhaps using a calculator, try other examples. Your aim in doing this is two-fold: to get an idea of what the answer to the question might be, and at the same time to develop a sense of why your answer might be correct.' p2

'Further careful specialising with an eye on the "why" rather than the "what" may lead to insight into what is really happening.' p4

This idea of looking at the particular but seeing the general was noted by Krutetskii (1976). He singles out 'on-the-spot' generalisation, which he describes as 'a maximal generalisation at once' (p240), as a feature of the work of mathematically very able students.

'The capable pupils we studied, when encountering a new kind of problem (problems they had not solved previously, in most cases) very often (but of course not always) interpreted and solved the first concrete problem of the type in a general form, as a general problem, discovering the "essence", singling out the main lines, and abstracting themselves from the external

aspect, from the particular, from the concrete data and numbers. Apparently, in a certain sense, this replaced for them the study of a general rule by which they were to operate. Thus, in solving the first concrete problem of a certain type, they - if one can so express it - were thereby solving all problems of that type.' (p247)

He also observed reluctance to deal with the concrete problem initially but a tendency to solve the problem generally and then specialise for the particular problem.

His emphasis is on the speed with which capable pupils were able to make generalisations. This speed has two aspects, first the short period of time required and secondly the small number of examples required. Pupils labelled as less capable were able to see the general structure of a problem after they had seen several particular problems of the same structure. To see the general structure from just one particular problem requires an act of generalisation which may be qualitatively different.

Davydov (1990) cites work by Mashbits (1963, 1965) which claims that students can be enabled to master the solution method to certain geometry problems, that is to generalise a method which they had seen applied to particular instances. These students 'ascertained the general structure of the solution method by analysing particular problems that were models, then rapidly and correctly applied the method to particular problems' (Davydov p329). They are contrasted with another group of students who worked through particular problems which were not specially selected as 'model' problems but varied in their concrete conditions and form of expression of the mathematical relationship. This second group were sometimes successful in forming a general method of solution but it took longer and the scope of application of their method was usually more limited than that of the first group. Davydov attributes these differences to the use of two different types of generalisation. He uses the terms 'empirical' and 'theoretical' to distinguish them. 'Empirical' generalisation is based on a large number of examples and proceeds by abstraction whereas 'theoretical' generalisation requires just one 'model' solution.

The notion of 'model' problems and solutions is a familiar one to most teachers of mathematics. A teacher 'working through an example' with her class can see the particular problem as an example of a class of problems. The teacher sees the general whilst working with the particular. She sees the features of this problem as 'placeholders' or variables which change to give a different particular instance of the same type of problem. Part of her task is to

enable the students to see the generality behind the particular. In order to do this she needs to move her attention constantly between the particular and the general.

These 'model' solutions can be seen as a form of generic example.

Acts of Generalisation

Using the two examples of tasks and the contexts specified by Krutetskii and Davydov as illustrations I want to indicate what I mean by some terms which I will use in connection with generalisation. The word *generalisation* itself I use to mean the expression of the general (and not its more technical meaning, that is the extension of the domain of validity of a mathematical statement). It encompasses both the process and the product, that is the act of expressing generality and the written or spoken product of that expression. In what follows I use the word 'generalisation' to refer to the act of generalisation by a particular student in a particular situation. The terms I will describe are not intended to be applied to a student independently of a particular piece of work or to a task independently of the student working on it. My uses of 'generalisation' encompass expressions of the general

- in developing concepts e.g. a function,
- in learning processes e.g. solving simultaneous linear equations, and
- in establishing results e.g. the cube of an odd number is odd.

These notes are not intended as definitions but as indications of which aspects of a generalisation I am emphasising.

An 'empirical' generalisation is one made primarily on the basis of results as opposed to process. For instance, in my first example above, concerning a question from an SMP textbook, a pupil who based their conviction (that the number of white tiles was always four greater than the number of grey tiles, or some equivalent statement) solely on the numbers in their table of results would be making an empirical generalisation. A purely empirical generalisation would ignore the shape and structure of the diagrams and the way in which the tiles were counted and focus only on the numbers in the table and a relationship between them.

I refer to an 'inductive' generalisation when I want to emphasise that it is the result of inductive rather than deductive reasoning. Inductive reasoning generalises beyond the domain in which the proposition is already established, that is it argues that if the proposition is true in every case so far tested then there is a good chance that it is true in every case. Propositions suggested by inductive reasoning are falsifiable by a counter example. Induction is

traditionally the domain of science and deduction of mathematics. (There is, however, a long history of induction in mathematics. Polya (1954) quotes a memoir of Euler in which he describes his discovery of a formula to determine $\sigma(n)$, the sum of the divisors of some positive integer n . Euler convinces himself and aims to convince a reader by inductive reasoning, that is on the basis of results rather than argument).

Deductive reasoning, on the other hand, reaches only conclusions which follow logically from its premises.

So in working on the SMP question a pupil who based their conviction on the results in the table would be arguing inductively, whereas one who based their conviction on a 'way of seeing' the diagrams would be arguing deductively. There is a good deal of overlap between my uses of 'empirical' and 'inductive' in this respect.

An 'on-the-spot' generalisation is one that is made on the basis of only one example. For instance, some of Mashbits' students apparently made such a generalisation based on just one 'model' solution.

Krutetskii uses this term to describe only correct generalisations, but that side of its meaning is not part of my use. In most cases it would be hard to have reasonable conviction based on inductive reasoning from only one example, but I will discuss incidents where students argue inductively from one case.

A 'structural' generalisation generalises a result from a single or several examples based on the generalisability of the process by which that result was obtained. For example, in my work on 'How Many Triangles?' I became convinced that the number of triangles having three vertices on the circle was $\binom{n}{3}$. I based my conviction on the generalisability of the way in which I had counted these triangles in the case $n = 5$. Such a generalisation, then, demands attention to those aspects of the process which are general and withdraws attention from those which are particular to the case under scrutiny. It also demands attention to the process and not just the result.


Some acts of generalisation could be described by more than one of these words. Some pairs of words, that is 'empirical' and 'structural', 'inductive' and 'on-the-spot' which have contrasting meanings, are not, nevertheless, mutually exclusive. Each draws attention to a different aspect of the act of generalisation.

Inductive Generalisation in Practice

Bases for Conviction

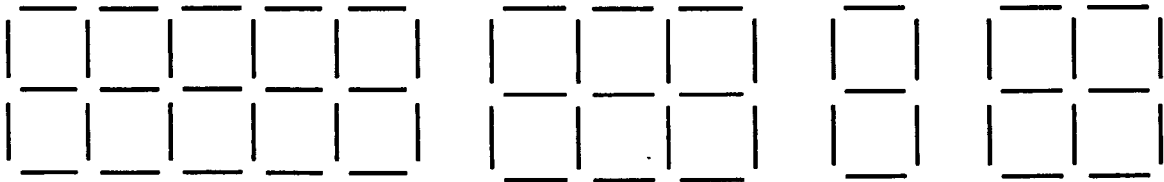
Conviction based on inductive reasoning is rarely based on inductive reasoning alone. For example, below I record my work on a problem called Rows and Columns

Problem C The picture below shows a rectangle made up of two rows of four columns and of squares outlined by matches. How many matches would be needed to make a rectangle with R rows and C columns?



You are invited to work on the problem yourself before reading further.

When I first worked on this problem, I decided to simplify by holding the number of rows constant. I held R as 2 and produced a series of diagrams such as these:



From my diagrams I produced a table of results as follows:

<u>No. of columns (C)</u>	<u>No. of matches (M)</u>
4	22
5	27
3	17
1	7
2	12

I saw that the results in my table fitted the rule $M = 5C + 2$.

My trust in this formula for all positive whole number values of C was based first on the results in my table.

Secondly, I was confident in it because it was of the form I was expecting. By this I mean that, first, I expected a relationship to exist between M and C . I was sure that I could easily predict the number of matches if I knew how many columns of squares were needed. Secondly I expected the relationship to be linear. This expectation was partly a matter of experience of similar problems

and partly a recognition that increasing the number of columns by one should always (no matter how many columns I had previously) result in the same increase in the number of matches.

Next I changed the value of R to 3 and, with the aid of one or two diagrams, convinced myself that M and C now fitted the rule $M = 7C + 3$.

Similarly, I found that, for $R = 4$, $M = 9C + 4$ and, for $R = 5$, $M = 11C + 5$.

When I went on to find formulae for M in terms of C for other values of R , I needed fewer diagrams and tabulated results each time. This was because my conviction about these formulae from sources other than my table of results was greater each time. Having established that linear relationships held for $R = 2$ and $R = 3$ I needed only two results in the case $R = 4$ in order to be convinced that I had the correct formula.

Now a pattern was emerging that suggested that a general rule was $M = (2R + 1)C + R$.

Again in moving from these separate formulae for different values of R to one which incorporated variations in R , I based my conviction first on the four formulae I had identified in the special cases $R = 2, 3, 4, 5$. But I had also the anticipation that such a general formula would exist, would be linear in R and in C and would be symmetrical with respect to the two variables.

Finally, looking at a diagram as a generic representative of the general, I was able to see that I could count the number of vertical and horizontal matches as follows:

there are $C + 1$ columns of vertical matches, each containing R matches
there are $R + 1$ rows of horizontal matches, each containing C matches
therefore there are altogether $(C + 1)R + (R + 1)C$ matches.

This line of argument confirmed the rule which I had arrived at empirically.

In other cases there are less tangible reasons for believing or doubting the truth of a general statement suggested by inductive reasoning. Here is an account from a lesson in June 94:

During a reporting-back session following an exploration of the absolute value function, Lorne makes the assertion that the graph of $y = |f(x)|$ is the same as the graph of $y = f(|x|)$ for every function f . I am unsure whether he is right and I try to think of counter-examples. I suggest that he plots $y = |2x + 1|$ and $y = 2|x| + 1$.

Lorne had considered five or six examples of functions in coming to this inductive conclusion. I had strong doubts about his conclusion, but not because

the number of examples considered was too small. Part of my doubt was because I felt that such a striking result would already have been known to me. Part of it was that I have an image of the graphs of modulus functions that involves points with undefined gradient, where the graph is 'reflected back on itself'. At first my feeling of unease was very vague. When I had had a moment to think it formed itself into a counter-example. Then I could see that Lorne's examples may have been sufficient in number but of 'the wrong kind'.

Another reason for recording this incident was that in retrospect I thought I might have suggested to Lorne that he look for counter-examples and allowed him to think about looking among a 'different kind' of function. In the moment of responding I did not consider this option because I was too occupied with my own doubts about the mathematics.

Whatever my intuition was which made me doubt the truth of Lorne's statement, there was no such doubt in Lorne's mind. I can account for this difference in two ways. First, he had less experience of the modulus function on which to draw. Secondly, he was less cautious of inductive reasoning. His schooling had often put him in the position of needing to trust conclusions from inductive reasoning in mathematics without considering the strength of other reasons for conviction.

However intuition should also be treated with caution as a basis for conviction. Rowland (Bills and Rowland 1996) speaks of intuition and conviction in connection with inductive reasoning. In speaking of his experience of coming to know that 'all quadrilaterals tessellate in the plane' he warns that intuition is a 'negative and dangerous reason for scepticism about the remarkable-but-unfamiliar'. This seemed such a remarkable property that he was sceptical of it on the grounds that he should surely have known about it before. In addition there were no instances in everyday life which would confirm this property of general quadrilaterals, floor and wall tiles being, almost without exception, rectangular.

For the experienced mathematician recourse to inductive reasoning is nearly always accompanied by other bases for conviction. The novice, however, may be encouraged to rely on inductive reasoning alone since the experience which might provide the basis for alternative reasoning is absent.

Inductive Generalisation in School

In the last ten to fifteen years a lot of attention has been paid in the United Kingdom to inductive generalisations in mathematics lessons. The Cockcroft Report's (Cockcroft 1982) endorsement of mathematical investigation together with the inclusion of compulsory coursework assessment at GCSE was the

catalyst for an enormous growth in the use of investigative tasks. Many of these tasks gave pupils the opportunity to generate empirical data and make an inductive generalisation concerning the relationship between two or more variables in the problem situation. In other words, the main aim in these tasks was to 'find the formula'. This approach allowed pupils to tackle problems which would previously have been considered far too difficult. The process of gathering data in particular cases and making an empirical generalisation from the patterns seen in the table of results was seen to be much more accessible for pupils than a deductive argument relating to the same situation. The empirical approach also had the potential to give pupils a real sense of discovery.

Dave Hewitt's article 'Train Spotters' Paradise' (Hewitt 1992) points out how activities aimed at encouraging children to generalise in this way can degenerate into 'pattern-spotting' exercises in which the essence or mathematical interest in a situation is lost in the abstraction of the number sequence or numerical relationship.

Moreover children involved in tasks which involve making a table of results can, in the process, lose sight of the purpose of the specialisation. It is particularly likely that they will do so if the process of gathering results is difficult for them. Getting a wrong result also impedes their ability to use the table to abstract a relationship between the two variables. So for some pupils the result can be a lesson in which they spend most of their time gathering inaccurate data for a purpose which remains unclear to them.

In such lessons the use of inductive reasoning is encouraged where it is not backed by other bases for conviction. The limitations of its role are not appreciated by the pupils.

I have a further concern about the widespread use of this approach to a problem.

In a lesson which I have described in chapter one students were working on the problem

Problem A For which values of k is $k(k - 1)x^2 + 2(k + 3)x + 2$ positive for all real values of x ?

Two pairs of students were taking an empirical approach. They had drawn up a table and were systematically varying the values of x and k and recording the value of the expression $k(k - 1)x^2 + 2(k + 3)x + 2$ for each pair of values of x and k . About half an hour after they had started work I had this conversation with them:

Liz Can you tell me what this is about?

Student 1 We're just putting in values for x and k and seeing what happens

Student 2 (*simultaneously*) trying different values of x .. er of k with a constant x

Liz Right. So you're

Student 2 and seeing what we get out

Liz you're working out a value for this expression with those two values

Student 1 Yeah

I came back to the same two boys later in the lesson

Liz What have you decided to do?

Student 2 Well, um, we're working out the formula for this one as well (*I interpreted him to mean that he was trying to find a formula for the expression in terms of x for a fixed value of k*)

Liz Right

Student 2 To try and . 'cos if we can work out the relationship between the two then we know that we won't need to follow that ... because so far this is just, er one, one answer further along than this

(they are looking at a table similar to the one below and comparing one row with another - it was not and is not clear to me what he meant by 'one answer further along')

	$x = -5$	-4	-3	-2	-1	0	1	2	3	4	5
$k = 0$	-28	-22	-16	-10	-4	2	8	14	20	26	32
1	-38	-30	-22	-14	-6	2	10	18	26	34	42
2	2	-6	-10	-10	-6	2	14	30	50	74	102

Liz Uh hm

Student 2 So we're just trying to distinguish whether or not it's worth pursuing the x , changing the x , or whether we should concentrate on changing the k s and leaving the x .

Liz Right . That decision I would have thought has more to do with what the question is asking you for than with what happens when you try it.

There are clues in their use of 'and seeing what happens' and 'and seeing what we get' that these boys had no clear idea of the end they are aiming for. They seem to have been, at least as a subsidiary aim, trying to obtain a formula for a relationship which has been stated in the question ('we're working out the formula for this one as well').

I suggest that they were following a well-worn track in producing a table of results and looking for patterns in it. Their attention was much more on the action of looking for a formula than on the process they had used to obtain the results or on what it was they had been asked to find out. They were exploring indiscriminately rather than seeking specific information.

For these two students, systematic specialising seems to be a strongly favoured strategy. They used it even in a situation in which their approach was (to me) plainly inappropriate.

My interpretations of some incidents, to be recounted later in this chapter, as involving empirical generalisation by students, owes something to my experience of this incident. It highlighted for me the extent to which these boys had adopted empirical investigation as a problem-solving strategy.

Use of Examples

In this section I bring my meanings for terms describing different types of generalisation and my concern at the use made of inductive reasoning to some accounts of the work of students involved in these processes.

How Many Examples?

During one of my meetings with the group of teachers we discussed a recording I had made of my conversation with a student who was working on finding the equation of a straight line. Towards the end of this discussion, one of the teachers, Kate, said:

'This has actually just shed some light on a conversation I had with my son. He was finding equations of straight lines through a point and I was saying to him "use $y - y_1 = m(x - x_1)$ " and he said "I've never heard of that before" and he wrote down for me $y = mx + mx_1 - y_1$ and I said "where did you get that?" (sorry " $- mx_1 + y_1$ " I'm getting it the wrong way round myself), and I said I had never seen it in that form before and he said "Well I did a lot of examples and I found that this pattern was working out" and it's the first time I've ever heard - I hadn't realised what I was hearing at the time - it's

the first time I've ever heard of somebody coming up with their own generalisation from doing a lot of numerical examples - and I now think my son's quite clever actually'

Although Kate expressed surprise at her son's generalisation I can recall similar occasions from my own teaching experience. For example, two boys, Paul and Kwok, working on the Cartesian equations of circles arrived at a general condition that the equation $x^2 + y^2 + ax + by + c = 0$ should represent a circle (this condition, that $\frac{1}{4}(a^2 + b^2) - c > 0$, comes from seeing that the quantity representing the radius must be real). They did this whilst working through an exercise which contained a large number of particular questions which asked 'Does the following equation represent a circle?'.

I find it instructive to note that Kate's son apparently came to his generalisation by spotting patterns in numbers ('Well I did lots of examples and I found that this pattern was working out'). He seems to have seen a relationship between the coefficients in the equations he derived and the gradients and coordinates of points that were given in the questions. That is, he saw that the y -intercept in the equations he derived was *minus the given gradient times the given x -coordinate plus the given y -coordinate*. Paul and Kwok did not do this but worked through, in the general case, the procedure they had been practising in several particular cases. They performed on the general equation $x^2 + y^2 + ax + by + c = 0$ the process which they had routinized on numerical examples. Kate's son's attention was on the pattern visible in his results, whereas Paul and Kwok's attention was on the process.,

What the two incidents have in common is that the students came to an algebraic expression of their own generalisation in the course of working on a lot of particular cases. There is however a subtle difference between the ways in which they arrived at these generalisations. I might describe the former as an 'empirical' or 'inductive' generalisation and the latter as a 'structural' generalisation.

Students' Use of Examples

Having made these distinctions I want to consider four further incidents in which I found these labels useful. The first is a conversation between me and one student, Frank. (The transcript extract is from 'Frank' lines 431 to 450)

I ask Frank to work on the question

Problem D Find the equation of a straight line which has gradient M and passes through the point (p, q) .

He says 'Let's try it with $y = mx + c$ ' and writes this down but then doesn't have a strategy for starting. He claims that he could do the question if he had values for M , p and q so I ask him to work with $M = 4$, $p = 2$ and $q = 3$.

He draws a sketch of the line in this case but then says he has forgotten the method for finding the equation. I take him through the steps of substituting known values into $y = mx + c$. We don't write anything more down but Frank works out a value for c in his head, saying

'Yes. It's 8 plus something equals 3. ... 8, it would be 8 minus 5. Yes. So that's got to be -5. So it's got to be $y = 4x - 5$.'

Next I ask him to work on the original question:

Liz: Uhmhm. Right, now the job that you've been given is to find the equation of a line which doesn't have them specified as numbers.

Frank: Yes. So that, $q = mp - c$. (*writes $q = mq - c$*)

Liz: No do you mean p there?

Frank: I do mean p , not q . .. p . I can't write either.

Liz: What did you use that equation for when you were doing the other one?

Frank: That?

Liz: Hmm.

Frank: I used .. p is x because it's the x -coordinate, q is y , because it's the y -coordinate (*writes $q = mp + c$*) m is the gradient and c is the constant. And because I didn't know the constant but because I knew the other ones I knew that $mp + c$ had to equal q , so I could just work out what c was.

Liz: Right. Well the same is true for this case.

Frank: Yes. So it would be $q = mp$. (*writes $q = (mp)$*) a bit of a shot in the dark ... 5 is what the two co-ordinates were when added together.

My intention in this interchange was that my example of the equation of a straight line going through (2, 3) and with gradient 4 should be the basis of a structural generalisation for Frank. I expected him to grasp the method and be able to apply it in the general case. His fourth utterance, which begins 'I used' indicates that he had grasped the method at some level. However, he did not go on, as I expected, to manipulate the equation $q = mp + c$ to give an expression

for c . Rather he went back to the numerical example we had done to look for a number pattern in the result. That is, he looked back to the given gradient, 4, and the given point, (2, 3), and to his answer $y = 4x - 5$ and spotted that $2 + 3$, the sum of the two coordinates, gave 5, the y -intercept in the equation (ignoring signs). Perhaps his hesitation over doing this ('a bit of a shot in the dark') was because he had only one example from which to generalise.

My first suggestion in explaining Frank's failure to do the algebraic manipulation is that this manipulation is not paralleled by the process he went through in the numerical example. His utterance at the time of performing the numerical manipulation ('Yes. It's 8 plus something equals 3. ... 8, it would be 8 minus 5. Yes. So that's got to be -5') makes clear that he is not seeing this process as $3 - 4 \times 2$, which is parallel to $q - mp$. This part of the numerical example, then, is not available as a pattern to follow in the general case.

I want also to suggest that empirical and structural generalisation are confusingly (to both teacher and student) mixed together in this incident. I expected and intended Frank to generalise from the structure of the numerical example. He, however, at least at this point in his work, referred back to it as data or empirical evidence for a relationship between the intercept in the final equation and the coordinates of the given point. A structural understanding of my example would have involved seeing that the specific numbers chosen, 4, 2 and 3, were arbitrary and understanding the relationship between them and the coefficients in the final equation in terms of the process used. An empirical view of my example would see only the given numbers 4, 2 and 3 and the final result $y = 4x - 5$, and would consider any connection between them as possibly generalisable.

My awareness of the problem situation told me that such a relationship between the coordinates of the point the line passes through and the y -intercept of the line was unlikely because the gradient of the line must be involved as well. I was sure that the relationship in this particular case was just a coincidence. Such an awareness was not available to Frank. He had no basis for conviction other than empirical evidence.

I was surprised by Frank's attempt to understand my example empirically because I expected him to generalise on the basis of structure and thought that this expectation would have been clear to him. However a few days later I recorded what I saw as a similar incident from a class lesson:

I am talking to the whole class about the way in which they derived the equation of a circle with radius 2 and centre (3, 5). I have written the equations $\sqrt{(x - 3)^2 + (y - 5)^2} = 2$ and $(x - 3)^2 + (y - 5)^2 = 4$ on the board. I ask 'where did the

3 the 4 and the 5 come from in this (the second) equation?' Trevor replies that the 4 is the diameter of the circle.

My question was intended to draw students' attention to the structure of the derivation of this particular equation, with the eventual aim that they would appreciate the form $(x - a)^2 + (y - b)^2 = r^2$ for the equation of a circle. I was expecting them to base an answer to my question on their recall of the procedure by which they had derived the equation. Trevor here seems to be making an empirical generalisation from one case, rather than recalling the procedure as I had hoped. His answer relies on seeing that 4 is twice the radius, rather than seeing that 4 is the radius squared and results from the squaring operation which was part of the process of obtaining the equation. I see his reaction as similar to that of Frank because he focused his attention on numerical patterns rather than structural relationships.

A common reaction from teachers with whom I have discussed the account of my work with Frank is to suggest that I should have done more numerical examples of a similar kind with Frank before asking him to work on the general case. This suggestion runs counter to the idea of my numerical example as a basis for an on-the-spot generalisation. The essence of such an example is that only one is required. However, it was felt by teachers reading the transcript that further examples would be useful. This could be seen as evidence of a pedagogic confusion between the roles of structural and empirical generalisation. Further examples would almost certainly have provided empirical evidence for Frank that his generalisation was incorrect, without necessarily shifting his attention from result to process. Or it may be that the teachers' experience suggested that students would more readily grasp a procedure if they had the opportunity to see it played out more than once.

Much earlier in the year my attention was drawn to this issue by an incident which I recorded as follows:

After some work on the remainder and factor theorems and on division of polynomials, Peter asks the class

Problem E Factorise

(1) $x^3 - 1$

(2) $x^3 - 8$

(3) $x^3 - a^3$

We both walk around the class looking at students' work.

After another few minutes Trevor asks for help. He has started (3) but has put $(x - a^3)$ as the linear factor. Frank and I explain to him why this is not going to work. My argument is along the lines that $f(a^3)$ is not zero. Frank says 'It's $f(a)$ because it's like the others'.

Frank had seen the connection between the first three expressions and was using his experience of the first two to solve the third. He preferred his explanation 'It's $f(a)$ because it's like the others', to mine ($f(a^3)$ is not zero). My interpretation of his words is that he sees the pattern in the solutions ($f(1) = 0$ when $f(x) = x^3 - 1$, $f(2) = 0$ when $f(x) = x^3 - 8$) as being more salient than the argument which he used in performing the first two factorisations (that is that $x - 1$ is a factor because $f(1) = 0$ and $x - 2$ is a factor because $f(2) = 0$). In my terms his emphasis was empirical rather than structural. This is in contrast to my attempt to draw attention back to the reasoning in the factorisation process.

This interpretation of Frank's thinking perhaps lays too great a burden of evidence on a few words which are open to different readings. However, the distinction between the modes of thinking remains important.

The fourth incident is from one of my meetings with teachers. It involves the question:

Problem F In how many ways can n 1 by 2 rectangles be arranged to form a 2 by n rectangle?

You are invited to work on the task yourself before reading on.

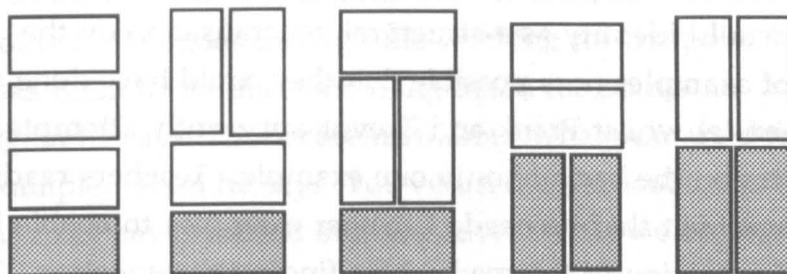
Three teachers were working on the rectangles problem:

Two teachers working together and one working on his own had independently come to the conclusion that the sequence of numbers of arrangements for increasing values of n was a sequence of Fibonacci numbers. Prompted to try to justify this conclusion, David, who was working alone, showed me how to obtain all the arrangements of four rectangles by adding two more rectangles to each of the arrangements of two, and one more rectangle to each of the arrangements of three. I asked him to show his demonstration to the other two.

David: If it's Fibonacci, for number four you add the two-combinations and three-combinations together

Valerie: Right

David: There are my two-combinations, three-combinations - so I just need to add one to each of those (*adding one further rectangle to each of the three-combinations*) and I need to add two to these (*adding two further rectangles to each of the two-combinations*) which if I add that way round I end up with all five combinations. hmm? So ..



Katherine: Why does - ?

David: They're the twos

Katherine: What happens if you add to the other side? Is it not possible to get any different ones?

David: I think that's going to be exactly the same results as if I'd added them on top. As long as I put these ones across and these ones down

Katherine: Because those two are the other way round - yes

David: Now - I haven't tried, but I guess three and four - I'm just assuming at the moment that it's just adding on - so that's four - and threes were - one, two, three (*laughter as David 'secretly' takes some more rods from the two women's work*) so I should be able to get all the combinations just going like that, that, that

After spending a few moments considering whether it mattered that the extra rectangles had been added at the bottom rather than the top, David pressed on to look at forming the five-combinations. I had asked David to give his demonstration because I thought it would serve alone to show that each term was the sum of the previous two. In fact both David and the two women seemed to want to look at another case, that of $n = 5$, in order to be convinced.

I suggest that these teachers were not looking for empirical evidence that their conjecture was correct. They had already seen that the sequence of numbers was a Fibonacci sequence. They were looking for confirmation of an argument, not of a result. In other words, they were looking through the particular to the general, rather than seeking statistical evidence. Nevertheless, they wanted to consider another example. The next example might well have been an

opportunity to rehearse their argument and look for flaws in it, rather than merely to gather empirical evidence.

On the basis of my review of literature on examples I might distinguish between structural and empirical generalisation by the number of examples needed. That is, empirical generalisation requires a number of examples whereas structural generalisation requires only one example. However, Paul and Kwok made what I would identify as a structural generalisation on the basis of a large number of examples (it is possible that they could have done as much after only one example) whilst Frank and Trevor apparently attempted an empirical generalisation on the basis of only one example. Teachers reading the conversation with Frank felt that he needed further examples to enable him to make a structural generalisation of the method for finding the equation. The three teachers working with the rectangles felt the need to look at a second example even though they were using a structural argument. In short, the number of examples used is not a reliable indicator of the distinction between empirical and structural generalisation. The way in which examples are perceived by the student as user may vary from the way in which they were intended, by the teacher, to be used. The difference between structural and empirical use of examples is a matter of the student's focus of attention.

This study of examples in use also suggests that a multiplicity of examples may be useful even in cases where they are used structurally. The distinction between the two kinds of generalisation may not be so easily made in practice as in theory.

Generic Example as a Teaching Technique

My review of literature allowed me to conclude that the generic example provides an alternative to the traditional view of concept formation by 'abstraction from many particulars'. A generic example is a vehicle for an on-the-spot, structural generalisation and acts as a stage between particular and general. Some have advocated the generic example as a teaching approach and others have observed generic abstraction as a stage in understanding.

My understanding of the theoretical arguments set out in chapter two concerning the role of examples in the formation of concepts made me sensitive to teachers' uses of examples in mathematics. I began to look for teachers' use of generic examples as a teaching technique. In this section I develop ideas about the 'generic example' as a teaching technique by referring to the practice of myself and other teachers and to various student responses which I observed as classroom incidents.

During November of 1993 I observed a lesson at School B in which Nigel was teaching some coordinate geometry techniques to the class. I noted:

Nigel is doing a presentation about finding the mid-point of a line joining two given points and the distance between them. He uses a generic example. He asks everyone to choose their own two points (in the positive quadrant) and find the mid-point and distance. He asks them to think about how they are doing it. He goes through the working for the point he has chosen and then asks them to do the same thing again for points not in the positive quadrant. Again he says 'Think carefully about what you are doing'. He goes through his example. Then he says 'You've used a particular method in doing that - you may not have realised but you have. What we are now going to do is generalise'. He draws a diagram showing two points (x_1, y_1) and (x_2, y_2) , two *general* points. By combination of asking and (mostly) telling he arrives at the formulae in terms of x_1, y_1, x_2 and y_2 for the mid-point and distance. They are left to do some straightforward exercises on the use of these formulae for homework.

Also in November 1993, I recorded the following about a lesson which I taught at School A.

I asked the students to write down the equation of a straight line with gradient 3 and intercept with the y -axis at $(0, 1)$. Next they had to write down the equation of a straight line with gradient 3 and intercept at $(0, -3)$, and then a general equation for a straight line with gradient 3. After that I showed them how to find the equation of a straight line with gradient 3 and going through the point $(2, 8)$. This process involved substituting the values 2 and 8 for x and y in the equation $y = 3x + c$. They did one more similar question themselves and then I split them into pairs and asked one of each pair to explain to the other in general terms the process by which they had found the equation of a straight line given the gradient and a particular point on the line.

Each of these accounts illustrates what I meant by the generic example as a teaching technique. The key features of my meaning at that time were

- (i) the teacher should demonstrate a process performed in a particular case but applicable to a wider range of cases
- (ii) the process should be one which the teacher intended students subsequently to be able to carry out
- (iii) the teacher should emphasise the generality of the process rather than the particular case she was working on

This third feature was illustrated in Nigel's lesson by his insistence that students 'Think carefully about what you are doing'. He also attempted to direct attention to the general case by asking all the students to work on their own particular cases, then on a second particular case, as well as seeing the case he chose to work on himself. In my lesson I attempted to focus on the process rather than the particular case by asking students to describe the process to each other without referring to the particular numbers I had used.

With this concept of the generic example as a teaching technique I began to explore the idea with teachers. In a meeting with teachers I used this narrative

I am going to define a transformation T as follows:

$$T:\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

I want to look at the effect of this transformation on a particular set of points, namely the line $y = 2x + 1$. I'm interested in whether the image set is a particular curve - that is whether there is an algebraic relationship between the x and y coordinates of any point in the image set. If I call these x and y coordinates X and Y , then:

$$X = 2x + 3y + 4$$

$$Y = x + 2y + 5$$

But if the point (x, y) lies on the line $y = 2x + 1$ then

$$X = 2x + 3(2x + 1) + 4 = 8x + 7$$

$$Y = x + 2(2x + 1) + 5 = 5x + 7$$

So

$$\begin{aligned} Y &= \frac{5}{8}(8x + 7) - \frac{5}{8} \times 7 + 7 \\ &= \frac{5}{8}(8x + 7) + \frac{21}{8} \\ &= \frac{5}{8}X + \frac{21}{8} \end{aligned}$$

I have found a linear relationship between X and Y and so I know that the image of the line $y = 2x + 1$ under the transformation T is another straight line.

Having presented this narrative to the meeting, I asked the teachers to

Problem G Prove that affine transformations map straight lines to straight lines.

(My own work on this task is described in chapter seven. A colleague describes his 'disastrous experience' in using this same approach with some pre-service teachers. The complexity of the algebraic manipulation in the general case proved very confusing for the students. This experience has something in

common with my own of attempting the proof in this way). Afterwards I asked them to consider how hearing the narrative had affected their work on the task.

During the subsequent discussion I put forward the view that my narrative portrays a procedure for establishing the nature of the algebraic connection between the coordinates of an image point under a transformation. My narrative treats a particular transformation acting on a particular set of points. I was asking the teachers to see the particular transformation as representative of a class of transformations and the particular set of points as representative of a class of sets of points, that is straight lines.

After the discussion with the teachers I gave a description of the way in which I had taught a class about finding the coordinates of a point which divides a given line segment in a given ratio. I said that I had chosen a particular line segment, from (1, 2) to (5, 7) and a particular ratio, 3 : 5. I had used a question and answer sequence to demonstrate a procedure for obtaining the coordinates of the new point. I stressed that I had used the particular example to talk about the general case. For example, I had referred to 'the x-coordinate of the first point' rather than 1, and I had not completed any calculations but left them in the form $1 + \frac{3}{3+5}(5 - 1)$. I left it open to the class whether they went on to derive a formula or simply to remember the method. In the meeting with teachers I used the phrase 'generic example' to refer to the teaching technique I was aiming to exemplify. I stressed the point given as feature (iii) above, that, in using the generic example as a teaching technique, the teacher needs to draw attention to the general rather than the particular.

With the teachers I also discussed the occasion, referred to above in the section 'Students' Use of Examples', on which I had worked with Frank who was expressing difficulties in working on this question: 'Find the equation of the line with gradient M which passes through the point (p, q) '. I had tried to help him by asking him first to find the equation of a line which had gradient 4 and went through the point (2, 3). I can see this as an attempt by me to use a generic example because I described the process used in the particular case in the hope that Frank would be able to use the same procedure in the general case.

Further instantiations of the generic example teaching technique have come to my attention through my discussions with colleagues. I offer these examples as a catalyst for you in recalling your own instances of teaching in a similar way. They also form a substitute for the discussion that might take place amongst a group reflecting on this issue. In fact most of these instantiations came to my attention by this means.

Rowland (personal communication) speaks of using a generic example to explain a proof in number theory. The multiplicative group formed from the residue classes modulo p is cyclic whenever p is prime. The general proof that this is the case can be paralleled very closely by a proof in a particular case, for example $p = 19$. Rowland's experience is that students much more readily understand the proof expressed in particular terms than that expressed in general.

In the Open University course, M101 'A Mathematics Foundation Course', a proof of Lagrange's Theorem is given which is based on the process of dividing a group into cosets of its sub-group. Prior to the proof students are invited to divide D_4 , the dihedral group of order 8, into cosets of one of its subgroups. The chosen subgroup has order two, so that the student's task is to divide the group into four sets each containing two elements. This is achieved by a process of forming cosets from elements not included in the cosets already formed.

In teaching this proof myself, I also invited students to make the division of a particular group before I expounded the proof itself. Having done so I was able to refer back to aspects of this example when dealing with the general ideas in the proof. For instance, in the proof I referred to the order of the group, the order of the subgroup and the number of distinct cosets, and was able to point back to the example to find these numbers as 8, 2 and 4. The process of proof, however, is not paralleled by the process of dividing into cosets. The proof, rather, takes this process as its starting point. The idea of a coset has been introduced as a tool for use in this proof, and the division of D_4 is offered as an example of a division into cosets.

Teachers' Responses

At the end of my meeting with the teachers, in which the above discussion of generic examples took place, I asked them to consider, over the next two weeks, whether they used any teaching approaches which were similar to the three I had demonstrated and described, so that we could share experiences the next time we met.

At our next meeting the following descriptions were given by five of the teachers:

- in teaching an upper sixth statistics class how to use tables to calculate probabilities associated with the Normal distribution, one teacher used particular numbers rather than statements like $P(Z < -a) = 1 - P(Z < a)$. For example, he would show them how to find $P(Z < -2)$ by obtaining $P(Z < 2)$ from the tables and calculating $1 - P(Z < 2)$.

- in working with a lower sixth class on solution of simple trigonometric equations, for example $2\sin x = \sqrt{3}$, another teacher encouraged his students to solve the equation in degrees, with which they felt comfortable, and convert the solutions into radians afterwards if necessary, rather than trying to work with radians immediately.
- a third teacher 'covered the blackboard with little examples' in order to encourage her lower sixth class to express the rule for differentiating a power function. She wrote, for example, $x^2 \rightarrow 2x$, $x^6 \rightarrow 6x^5$.
- a fourth teacher said that she had previously used precisely the method I described for teaching about dividing a line in a given ratio.
- in a year ten class a fifth teacher suggested that students who could not see how to manipulate the algebraic expression to solve, for example, $12 = \frac{2}{x'}$ should substitute numbers which make it easy to see the answer, in this case perhaps $5 = \frac{15}{x}$. Then, seeing that the answer could be obtained as $15 \div 5$, they might apply the same structure to the original equation to get $x = 2 \div 12$.

Some of these descriptions I did not see as fitting with what I meant by 'similar teaching approaches', but rather than discard them I want to use them as a means of exploring what the teachers perceived as the relevant features of my examples.

First, one primary aim in each of the examples I described to the teachers was to assist students in expressing the general case algebraically.

In the case of the affine transformation I was asking for a general proof in the expectation that it would involve algebraic expression of a general affine transformation and a general straight line. In the case of the division of the line segment my main aim was to enable students to solve similar problems in the future, but a subsidiary aim was that they might express the method for calculating the coordinates of the point as an algebraic formula. In the case of Frank working on the problem 'Find the equation of the line with gradient M which passes through the point (p, q) ', the algebraic expression of generality was an overt aim.

In all of the teachers' examples however, solving similar problems in the future was a paramount aim, and only in the differentiation example was an algebraic formulation explicitly sought.

Secondly, in the examples I described, the eventual general task was first made more particular, whereas in the case of the trigonometric equations and the year

ten equations the task was first made not more particular, but simpler or more familiar.

Thirdly, my aim in each of the three examples was that students should understand the reasoning behind the procedure used in the particular case and be able to transfer that reasoning to the general case. In contrast, the differentiation case asked students to see the pattern in the particular examples presented to them but not to know how these particular results arose. Another way of expressing this difference is to say that my examples asked students to make a structural generalisation whereas the differentiation case appealed to students' inductive or empirical reasoning.

I can summarise these distinctions by saying that whereas my emphasis was on helping students to move from particular to general, and to demonstrate that they have done so by expressing the general case, the teachers' emphasis was on enabling the students to operate competently with the particular. This difference in emphases does not necessarily correspond to a difference in the way students perceived the tasks. In particular, whereas my aim was for students to understand the structure of the solution procedure in the particular case and to transfer the structure to the general case, it may be that students would use the particular case as a template, merely substituting the general expressions for the particular without assimilating the method.

In looking for similarities between my examples and the teachers' examples, I find that we are all expecting students to see an essential 'sameness' between examples. In each of my three examples I am expecting students to see that the general case is 'essentially the same' as the particular case that I have worked through. In its most extreme form this awareness allows students to see, for example, that they don't need to work through the proof that all affine transformations map straight lines to straight lines because the working would be 'essentially the same' as that which I had already done. This appreciation of sameness is an expert awareness which legitimates the use of 'similarly' and 'by symmetry' in proofs. It could also rest happily within Arcavi's list of the constituents of 'symbol sense' (Arcavi 1994). This awareness of sameness goes beyond the use of a generic example as a template. Frank (see below, 'Seeing the General in the Particular') was able to develop a procedure to factorise $x^3 - 1$ which he then used as a template in order to factorise $x^3 - 8$ and $x^3 - a^3$. This procedure was to use the factor theorem to find the linear factor and long division to find the quadratic factor. But he was not aware of the summarising role of the general $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$. He was able to experience the procedure of factorisation as generic but not to experience the form as an expression of a generalisation.

In the teachers' examples there is a similar need for an appreciation of sameness on the part of the students. The approach to teaching the normal distribution relies on students seeing the sameness of the method for finding $P(Z < -3)$ and $P(Z < -2)$. The use of degrees instead of radians relies on students seeing that the same solution method works for both. The technique for eliciting the method for differentiating a power function relies on students seeing that all the examples are of 'the same thing'. Finally the technique used with year ten students working on equations relies on an appreciation that $12 = \frac{2}{x}$ and $5 = \frac{15}{x}$ are the same type of equation, in the sense that what is permissible for one is permissible for the other and that a solution method for one will work for the other.

In summary, my concern with the move from the particular to the general was not seen as the most important feature of the 'generic example' by the teachers. However the requirement for students to appreciate the sameness of two or more examples was a feature common to the situations we described as 'generic examples'.

The incidents I have noted during the course of my research lead me to believe that a distinction between structural and empirical generalisation is valid and useful. However, it is not possible to conclude that, in contrast to empirical generalisations, structural generalisations can necessarily be made on the basis of only one example. The generic example as a teaching technique, as I have used the term, appeals to students' capacity for structural generalisation, but is not necessarily at its most effective when used singly. Generalisation based on a 'generic example' need not be 'on-the-spot'.

Examplehood

Here I look at two occasions when the contrast between students' and teachers' experiences of teachers' examples became apparent.

Nigel does an example of adding algebraic fractions on the board. The task is to simplify $x + \frac{1}{x}$. This is rewritten as $\frac{1}{x} + \frac{x}{1}$ and then a common denominator of x is chosen, giving $\frac{x^2}{x} + \frac{1}{x} = \frac{x^2 + 1}{x}$. At this point Kris wants to cancel the x with the x^2 to leave $x + 1$. Nigel tries to explain why this cannot be done by giving the example of $\frac{4 + 12}{8}$, and saying that 4 and 8 cannot be cancelled. But Kris rejects this example because 4 is not 8 squared.

Here Nigel's example is not an example for Kris. Kris is more concerned with the differences between the original example and Nigel's example than by their sameness.

The teacher uses examples in the hope that students will connect the particular case to the general case which the teacher sees. However there is always a danger that the student does not perceive what the example is an example of in the way that the teacher intended.

A colleague and I were visiting a sixth form class which was working on differentiation. The students had been introduced to $y = |x^2 - 9|$ as a non-differentiable function. The teacher offered an explanation of why this function was non-differentiable. My colleague then asked the class for other examples of non-differentiable functions. There was a silence until somebody offered, by gesticulation and then drawing rather than by naming, the graph of $x = |y^2 - 9|$. The generality represented for the teacher by her example had not apparently been perceived by the students.

These students were following a published scheme with their teacher. The chapter on differentiability suggested several examples of non-differentiable functions for consideration. The teacher had decided that, given the constraints of time, she would draw only one example to their attention. One interpretation of her actions is that she expected the students to be able to make a generic generalisation where the scheme expected an abstraction from many examples.

Moving from Particular to General

There is a paradox implied in any suggestion that the general may be seen in the particular. I must have an appreciation of what the general case would look like before I can see it in the particular, but I cannot move to the general except through the particular. The latter contention is disputed by Davydov and Krutetskii in their assertion that a direct grasp of the general is possible.

This paradox is a special case of that described by Bereiter in Bereiter (1985). He describes the 'learning paradox' as follows

'To put it most simply, the paradox is that if one tries to account for learning by means of mental actions carried out by the learner, then it is necessary to attribute to the learner a prior cognitive structure that is as advanced or complex as the one to be acquired' (p201).

Bereiter suggests ten ways in which the 'bootstrapping' necessary to overcome this paradox might occur. One of these is the use of Learning Support Systems. He says

'.. the function of these various learning support systems is to focus the learner's attention on significant variables and to provide simplified formats for the complex interactions that are eventually to be learned and internalised' (p 212).

One obvious Learning Support System is school teaching. I suggest that the generic example teaching technique is a means of 'focusing the learner's attention on significant variables'. Freudenthal describes the paradigm as a means of 'apprehending', that is achieving a direct grasp of the general situation. He, like Krutetskii, describes pupils seeing the general 'all at once', in one particular. However, Bereiter claims that 'appeal to learning support systems does not dispose of the learning paradox' (p212). In the same way acknowledgement of the generic example as a teaching technique does not solve the particular-general paradox which I have posed. It merely asserts that it is possible to see the general in the particular without having a prior appreciation of the general situation.

This, then is a substantial claim. My experiences point as much to the failure of the teacher's intention to communicate generality through a generic example as to its success. The choice of example is clearly crucial to the success of the communication. For instance, in my description of my work on Problem B, 'How Many Triangles?', I say that, in working on the case $n = 4$, I failed to see how the triangles with only one or two vertices on the circle could be counted in the general case. The particular features of the $n = 4$ case made it difficult to see this case as generic. In this sense $n = 4$ fails to meet the demands of Harel and Tall's 'parallel principle' (see chapter two). I was later successful in making a generalisation by going on to consider other cases, but not simply by treating these cases empirically or by reasoning inductively.

I began this chapter by distinguishing between generalisation based on many examples and generalisation based on one, generic, example or paradigm. As I have looked at incidents from my own experience in the light of this distinction, I have found it more instructive to distinguish between empirical and structural generalisation than between generalisation relying on many and on one example.

Kate's son and Paul and Kwok each made a generalisation from many examples but they used those examples in very different ways. Frank and Trevor both attempted to make empirical generalisations when they had only one example to work from. David and his colleagues felt the need of further examples even though the one example they had looked at contained everything they needed to construct the general argument.

I suggest that in each of these three cases I find out more about the thought processes of the individuals by asking whether they were making an empirical or a structural generalisation than by asking whether they generalised from one or many examples.

Having looked at students and teachers forming generalisations, I now turn my attention to the expression of generality.

Expressing Generality

The expression of generality is widely recognised as one of the powers of mathematics. Polya (1945) and Mason et al (1985), amongst others, place it at the heart of mathematics. However Pozzi (1993a) suggests that 'A' level students are rarely called upon to express generality.

The forms which are frequently used at 'A' level, for example

$$y = ax^2 + bx + c,$$

$$\frac{d}{dx}(u(x)v(x)) = u \frac{dv}{dx} + \frac{du}{dx}v,$$

$$\left(\frac{\lambda x_2 + \mu x_1}{\lambda + \mu}, \frac{\lambda y_2 + \mu y_1}{\lambda + \mu} \right)$$

are usually presented as generalisations or formulae. The students' task is then to 'use the formula' in a particular case, that is to specialise. Derivations of formulae are often presented by the teacher or the text book but not expected of the students. But to use a formula is not necessarily to experience the generality behind it. Without this experience of generality, I suggest, it is much more difficult for students to deal with generality in slightly different or completely novel situations.

Using second variables in expressions of generality

Below I give two accounts of situations in which students have been asked to express generality and have found difficulty in doing so.

At a summer school for the Open University course M101 'Mathematics: A Foundation Course' students are asked to find the flaw(s) in the following proof:

Claim The set of matrices

$$S = \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} : x, y \in \mathbb{R}, x^2 - y^2 = 1 \right\}$$

is closed under matrix multiplication.

'Proof' Let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ so that $A \in S$.

$$\text{Then } A \times A = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{bmatrix}.$$

Let $a^2 + b^2 = c$ and let $2ab = d$.

Then $A \times A = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$, which belongs to S .

So S is closed under matrix multiplication.

As was foreseen by the question writers, many students do not immediately notice that this expression of two general elements of S is not sufficiently general for a proof of this result.

In my second example a colleague asked her students (on an undergraduate course in mathematics education) to prove that if they took any number, reversed its digits and took the smaller number from the larger, the result would be a multiple of nine. In constructing a proof the students were encouraged to use n to stand for the number of digits in the original number. The proof requires that the difference between the two numbers is expressed as a sum of terms, each of which is divisible by nine. Since n is variable I cannot look at each term individually and I must show that a *general* term in the expression is divisible by nine. My colleague reports that at this stage her students saw no need to use a different letter to label the general term, but wanted to use n again.

Each of these two incidents involves a reluctance on the part of students to use a second letter (or pair of letters) in their expression of some generality. In the case of the summer school students, the second pair should play an entirely symmetrical role with the first. The students (mostly) are happy with a and b in both roles. Perhaps their reasoning is, even if unconsciously, 'Since a and b could be any real numbers, I can use them to stand for two (perhaps) different pairs'. The form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is seen as a whole, with a and b representing locations rather than particular numbers. There is then no difficulty in the same form later representing a different matrix. The majority readily accept, when it is pointed out to them, that a different pair of letters is needed for the second matrix

In the case of my colleague's students, however, the role of the second letter was quite different from that of the first. While one represented the number of digits and was fixed once the number whose digits were to be reversed had been chosen, the second was a generalised whole number which must continue to stand for any of a range of positive whole numbers, even when the number in question had been identified. Varying the quantity represented by the first

number (n), represented changing the number of digits in the given number, whilst varying the second represented moving from one term to another in the expression for the difference. The order in which these two variables are varied makes a crucial difference to the way in which the situation is viewed (see 'Order of Variation', chapter six).

In this situation the students had been asked to find their own way of expressing the generality. In these circumstances, to conceive of a second and different role for a second variable seems to be beyond their prior experience and substantially more difficult than choosing to use a letter to stand for the first variable (the number of digits). This is despite their considerable experience of using formulas and expressions involving second variables.

A teaching technique - PPG

In my work with the two sixth form classes I developed a teaching ploy to encourage, amongst other things, practice in expressing generality. The name I gave to this ploy was 'Particular, Peculiar, General' abbreviated to PPG. First I would ask students to write down, for example 'A particular number which leaves a remainder of 3 when divided by 7'. Typical responses would be 10, 17, 24. Next I would ask for 'A peculiar number which leaves a remainder of 3 when divided by 7'. Here I got responses like 703, 39 826, 7 773, 70 003. Finally 'A general number which leaves a remainder of 3 when divided by 7' solicited responses equivalent to $7x + 3$.

The first of these three injunctions is intended as an orienting device. It gives the students time to consider what kind of numbers have that property. The second injunction is intended to move towards the general by asking for something beyond the immediate experience. For instance, in the example above, I wanted students to go beyond the 'seven times table' so that the number they chose was more obviously 'seven times something plus three' rather than 'something in the seven times table plus three'. This might then enable them to move on to 'seven times x plus three' in response to the third request.

To begin with the students found the third stage very difficult. They became more comfortable with the task as they had more experience of being asked for an expression of generality.

As we progressed I made the task suitable for the topic we were or had been working on as well as increasing the complexity. The following are examples which I used:

- an even number

- a number which leaves remainder 1 on division by 100
- a number which leaves remainder 1 on division by n
- two numbers whose mean is 10
- two numbers whose mean is n
- a function whose derivative is 3
- a number raised to a power which is twice the original number
- the equation of a straight line whose intercept is twice its gradient
- the sides of a rectangle whose area is 24
- an angle whose sine is one half
- a parabola whose vertex is at the origin

Two of these examples, 'a number which leaves remainder 1 on division by n ' and 'two numbers whose mean is n ', require the introduction of a second letter in response to the request for the general. I used the first of these with the class at school A a week after having asked them 'a particular number which leaves a remainder of 3 when divided by 7' and 'a number which leaves remainder 1 on division by 100'. All but one in the class made an appropriate generalisation.

I used the second with the class in school B two weeks after using PPG with them for the first time. On that occasion I had asked for 'two numbers whose mean is 10'. When asked for 'two numbers whose mean is n ' only two of the class were able to generalise correctly. Of the others, four gave answers which involved only n and six answered ' x and $n - x$ '.

I account for the difference between the two classes in two ways. The first way is by noting that class A had previously had two similar but less general examples to work on, whereas class B had had only one. The second is by claiming that class B were less willing to try something they were not confident in and less trusting of somebody who asked them to do so.

One of the teachers who attended a series of meetings with me adapted the idea of PPG to teach about integration. He told me about it as follows:

David Um I didn't tell them it was integration, I just put the title 'Particular, Peculiar and General', which at least got their attention, and we'd been doing differentiation so I put I think it was the one I put 'If $\frac{dy}{dx} = 4x$ what does y equal?' and having looked sort of a bit blankly quite quickly thought they knew the answer. I told them to actually write it down. A few minutes later, once they'd found their pens and everything

Liz Thought they knew the answer?

David Well, yes, they thought they knew the answer, after a while they were sitting there with smug grins on their face and so I just said 'Right, er I want a different answer and I want you all to come up with different ones, not just one other, I want you to, when you realise how to do a different answer to this to make sure that you put one that no-one else will think of'.

Liz Uhuh, that's a nice way of putting it

David And there was lots of blank looks then and it wasn't the ones I was expecting that suddenly popped up 'Oh!' and luckily it was quite a while before somebody said 'Oh, you can just add anything!', so there was quite a few people actually discovering it for themselves by racking their brains and realising. And so they did the 'peculiar' thing. And I said 'Right, we've done a particular one, we've done a peculiar one, how would you cope with a general one?' And so they said 'oh, you can put anything' and so I said 'Well put down something for anything' and so they all shoved a letter down

He went on to describe how he had discussed with them which letters could and could not be used. He told me that the students had decided that they could not use e, i, j or any other letter with a special meaning in another context. I asked if anybody had said that they couldn't use x and he said 'Oh yeah, they said x and y and things like that'. David took it for granted that x and y could not be used and was most interested in mention of letters with special constant meanings, whereas for me the exclusion of x and y was much more interesting. My concern was with the students' recognition that the 'constant of integration' was fundamentally different from the 'variable', x . David was more concerned with the difference between the 'constant of integration' and a 'real' constant, that is one which has a fixed value. The issue of students' choice of letters which is raised here is dealt with more fully in chapter six.

On this occasion the students were supported in expressing a generalisation which they, as a class, had made.

Students expressing generality

On another occasion I encouraged the class at school A to express some generalisations concerning basic differentiation which had been established inductively by the class. They had obtained gradient functions for a variety of simple polynomial functions by zooming in on points of the curve on a graphic calculator. We achieved a bank of results by pooling resources. Once all of the results were gathered I led a conversation in which I invited the students to notice some general rules. For example

Liz I'm thinking for instance particularly of (e), can you make any sort of general statement from what you found out from (e)? ((e) asked students to find the gradient function for $y = \frac{1}{2}x^3$)

Sam: It's the same. You multiply $\frac{1}{2}$ by 3. You say 3 times $\frac{1}{2}$ is $\frac{3}{2}$ then (inaudible)

Liz: What do you mean by its the same?

Sam: Well because you're taking the old power, and multiplying the power of x by it (inaudible) and then the power reduces by one

Once a number of these 'rules' had been established I asked the students to write them down 'in a more succinct way', giving the 'usual rule',

$\frac{d}{dx}(x^n) = nx^{n-1}$, which was in their text book, as an example. Some of their responses follow:

Trevor: $\frac{d}{dx}$.. bracket cx^n equals cnx^{n-1}

Lorne: $\frac{d}{dx}(x^n + x^m) = nx^{n-1} + mx^{m-1}$

Kevin: $\frac{d}{dx}(a) = 0$

Frank: Umm, $\frac{d}{dx}(kx) = k$.

These students demonstrated a familiarity with expressing generality in their facility to express these differentiation rules algebraically. They also showed that they were conversant with the conventions concerning the use of particular letters. For instance, Kevin and Frank used a and k conventionally

by my example, and m to stand for another power. He used (alphabetically) neighbouring letters in similar roles. Lorne also showed that he realised the need for two different letters in addition to x .

Teachers expressing generality

Such expression of generality becomes 'second nature' to the experienced mathematician, so that the skill they have acquired and the decisions they make concerning which letter to use become invisible to them. The teacher with whom I worked at school A noticed this in himself. After I had spoken to him about some students who had had difficulty in working with a 'general point' he said that the placement of the point was so obvious to him that he had not thought of the students' potential difficulty.

Daniel, who was a member of one of the teachers' groups with whom I met was discussing a problem involving the equation $y = a \sec x$. He said

'And I need to spend more time with - now I'm appreciating more of the fact that I quite naturally will use these - what appear to be variables but which are in fact just generalised constants - I call them placeholders - then although I do that naturally, it is confusing for them.'

My presence was a catalyst to Daniel's becoming aware of the difficulties that can be entailed for students in expressions of generality.

For the teachers, expressions of generality will be true *expressions* of some general notion. For the student they may be inappropriately described as *expressions*, since the student is not aware of the generality which they purport to express. One response to this awareness is for teachers to make more opportunities for students to express generality for themselves, rather than expecting students simply to accept expressions provided by the teacher.

Expressions of generality which involve more than one variable give rise to problems over and above the expression of generality *per se*. My use of the technique 'Particular, Peculiar, General' suggests that students can improve their skills in expressing generality by practice, that is by repeated exposure to increasingly complex generalities to express.

Expressing generality in a diagram

The subject area with which I have been concerned falls very largely into the realm of algebra. Much of the teaching and learning I have taken part in has been in the area of coordinate geometry but the methods by which this subject is taught in British schools are predominantly algebraic. Typically a diagram is drawn to represent the relationship between the curves, lines and points in question and then that relationship is expressed algebraically.

drawn to represent the relationship between the curves, lines and points in question and then that relationship is expressed algebraically.

In these circumstances I have noticed a difference between the relationship between particular and general in algebraic and in geometric situations.

A 'general' point

I begin with an account of a lesson at school A in December 1993.

Peter spends today's lesson going through the test which the class did last Thursday.

Question 5 has caused difficulties for everybody (nobody has got any marks for it) and reads

Problem H A point P , coordinates (a, b) is equidistant from the x -axis and the point $(3, 2)$. Find a relationship connecting a and b .

Peter asks them to write down the two distances involved i.e. the distance from (a, b) to $(3, 2)$ and the distance from (a, b) to the x -axis.

As they work on this I speak to Kevin, Frank, Nuria and Trevor, in particular to Kevin. He has drawn a diagram showing the position of $(3, 2)$ and the axes but has not marked on (a, b) . I ask him where he is going to put (a, b) and he says that he does not know. I say that it is a general point and can go anywhere. He is trying to place it so that it is equidistant from $(3, 2)$ and the x -axis. I assure him that this is not necessary and suggest a general area in which he should place it.

Meanwhile Trevor has placed his (a, b) on the x -axis. I suggest that he put it somewhere else and he says 'But you said it could go anywhere!'

When they have got suitable diagrams I ask them to find the distance between the two points. Nobody mentions the formula for finding the length of a line joining two given points so I ask them, by pointing at the diagram, about the horizontal and vertical distances from the axes of the two points $(3, 2)$ and (a, b) . Kevin knows that the horizontal distance of $(3, 2)$ from the y -axis is 3. When I ask him the horizontal distance from the y -axis to (a, b) (by pointing) he says 'It looks about 1'. I say 'I don't want "it looks about ..", I want an exact distance'. He says, 'Okay, exactly 1 then!'

I understand Kevin's initial behaviour as being similar to ~~that~~ of a pupil who wants to know the value of the letter (letter evaluation is identified as one of the most prevalent early reactions to algebra tasks by Küchemann (1981)). I learn from Kevin's later answers concerning distances on the diagram that he

the diagram a coordinate system is implied and the placing of (a, b) does give values to a and b). He was cautious about where to place it, in particular trying to place it so that the condition expressed in the question was true. My purpose in asking the students to draw a diagram did not depend on the point being placed on the diagram in such a way that the conditions stated were true for that point. I expected them to use the diagram to find a way of expressing certain distances in terms of a and b . It was not important for me that the diagram gave the appearance that the distances were equal. However for Kevin the positioning of (a, b) was very important. He wanted the point to have the appearance of equidistance. Moreover I think it likely that he was looking for the one correct location for (a, b) . My evidence for this assertion is that Kevin gave me a numerical answer when I asked him about the distance from (a, b) to the y -axis. The implication is that the placement of (a, b) assigned values to a and b rather than leaving them as general. If placing (a, b) assigns numerical values to a and b then it matters where I put it. If it is a general point whose distances from the given point and line will be expressed generally, then its placement does not matter.

However the geometric nature of this problem raises new issues. Although I have claimed that placing (a, b) on the diagram is equivalent to choosing values for a and b , I do not suggest that, when Kevin finally marked the point on the diagram, he thought of his action in this way. Because I can place a point on a diagram without being explicitly aware of the numerical values of the coordinates of its position, those values can be thought of as general in a way in which numerical values chosen for letters cannot. Replacing a letter by a number makes the particular nature of the number visible. Placing a point on a diagram can be done without acknowledging the particularity of its position. In this way I can 'pretend' that the point is not particular. The difference between particular and general is much more visible in using numbers rather than letters than it is in placing a point somewhere in particular rather than placing it nowhere in particular!

This necessary pretence (that the placement of a point is not particular) makes it much more difficult for the learner to appreciate the difference between placing a particular point and placing a general point. The more experienced mathematician can cope with this feature of geometry, that a representation which is necessarily particular can be treated as general. She can distinguish those properties of the point which would be true of any point from those which are consequences of its particular position.

One interpretation of Trevor's behaviour is that he had sufficient confidence in my statement that (a, b) could go anywhere to place it somewhere which was

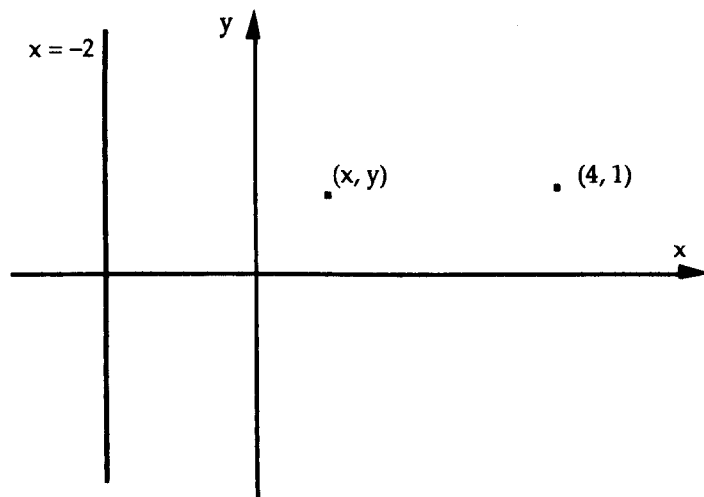
convenient for the subsequent working. What is clear is that he failed to distinguish between general and particular properties of (a, b) . His literal understanding of my statement suggests that he has mistaken the nature of the freedom (as in freedom and constraint) of this point. He thought that he could choose any particular values for a and b , whereas almost the opposite is true, that is he could not choose any particular value of a or b , in order that no potential values are excluded. His placing of the point on the axis makes impossible the pretence that the point is general.

In June 1994 I returned to the topic of locus with this class. I recorded the following after the lesson:

A little later Trevor is working on question 1 from the exercise which I have set. It reads

Problem 1 Find the Cartesian equation of the set of points P , if P is equidistant from the point $(4, 1)$ and the line $x = -2$.

He says he does not know how to do it so I look at his diagram. He has marked the axes, the line $x = -2$ and the point $(4, 1)$. I ask him to place the point (x, y) on his diagram, which he does, taking care to place it so that it is equidistant from the point and the line. The point is positioned at roughly $(1, 1)$.



I point to (x, y) and its perpendicular distances from the two axes and I say 'this is the point (x, y) , so this distance is x and this distance is y .' Pointing to the distance between the line $x = -2$ and the y -axis I say 'this distance is 2'. Trevor nods his agreement on each occasion. Pointing to the perpendicular distance from $x = -2$ to (x, y) , I say, 'what is this distance?' Trevor replies 'It's 3'. I talk about (x, y) being a 'general point' and he finally volunteers ' $x + 2$ '.

On this occasion I had placed more emphasis on the geometrical aspects of locus in what I had asked students to do. Before we derived the equation of a parabola from the equidistance condition I had asked them to find some points on the diagram which satisfied the condition and see what kind of shape the locus had. Subsequently we derived the equation of a parabola and confirmed that this fitted with our initial ideas on what the curve should look like.

This experience probably contributed to Trevor's desire to place the point (x, y) so that it was equidistant from the point and line (and to my decision as teacher not to suggest that he do otherwise). The definite nature of his answer, 'It's 3', to my question about distance indicates that this was his intention. On the earlier occasion Kevin answered 'It's about 1' to a similar question. Kevin seems to have placed the point on the diagram before thinking about the value of its coordinates, whereas Trevor deliberately placed (x, y) at the point representing particular choices of coordinates.

The question of the nature of the freedom of this point is raised again. Trevor thought that he could choose any particular point which satisfied the condition, when in fact what was needed was to choose a representative position which possessed only those characteristics necessarily possessed by a point which fulfilled the conditions of the locus. This is equivalent to placing the point in such a way that the pretence of generality can be maintained. In the light of this my earlier injunction that the point (x, y) could go anywhere seems particularly unhelpful.

In May 1994 I had an individual conversation with Eddie (the transcript extract is from 'Eddie', lines 60 to 79). I set him this problem

Problem J Find, in terms of a and b , the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$.

Eddie worked out the gradient of the given line and then seemed unsure how to continue.

Liz Do you know what the foot of the perpendicular means? Do you know what that term means?

Eddie No I don't to be honest

Liz Well it means, if .. you've got this line and you've got this point. You draw a line from the point to the other line, um and it meets it at a right angle (*Eddie draws a diagram showing two lines intersecting at right angles during this speech*)

Eddie Yeah

Liz Can you put the point on?

Eddie (a, b) ?

Liz The point (a, b) yeah

Eddie It should be there then (*he marks the point (a, b) at the intersection of the two lines*) where the lines cross

Liz Which was your original line?

Eddie The $x + 2y - 4$ line

Liz Yeah, is that this one? (*I point to one of the intersecting lines*)

Eddie Yeah

Liz Okay. ... Do you know that this point is definitely on that line?

Eddie .. Well a and b could be any . variable, can't they, so it must be one point on the line

Liz ... well .. no, not necessarily, because as you say a and b could be anything

Eddie Yeah, they can, but let's presume they can be any number, which could fall into that category. I mean if they were five million and a half they wouldn't fit into that equation obviously but as they are variables you can just presume they will .. well, that's how I would anyway!

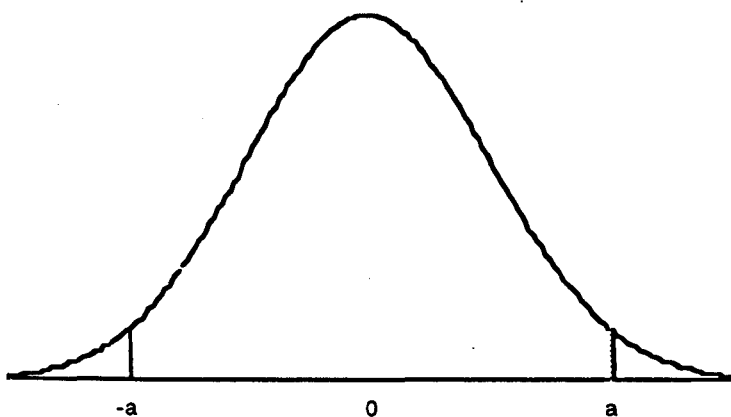
Eddie's view of the role of (a, b) in this question was very different from my own. He seems to be saying that he was at liberty to make choices about the values of a and b because they were variables ('as they are variables you can just presume they will'). In this his behaviour is similar to Trevor's, who wanted to put the 'general' point in a particular place of his own choosing.

Eddie's placing of (a, b) at the point of intersection suggests that he was thinking of a and b as unknowns rather than variables. He started from a position of 'knowing' something about (a, b) (that it lay at the intersection of the two lines) and wanting to find the values of a and b . This suggestion is reinforced by what happened later in the interview. Eddie's first attempt at an answer to the question was to give expressions for a and b in terms of other letters. My intervention in asking him 'Do you know that this point is definitely on that line?' forced him to try to justify his placing of the point at the intersection, which he did in terms of his own choice.

I see a connection between students' skills in using 'general points' and their skills in expressing generality. Both are enhanced by an awareness of freedom

and constraint. Decisions concerning the use of literal symbols in expressing generality rely on an appreciation of the number of degrees of freedom available. A diagram which involves a 'general point' is very often produced as an aid to expressing the freedom and constraint which exist in a mathematical situation.

During a meeting with teachers, which is described more fully in the section 'Generic example as a teaching technique' earlier in this chapter, teaching the use of tables to find Normal probabilities was discussed. One teacher described how he would explain, for instance, that $P(Z < -2) = 1 - P(Z < 2)$, rather than, more generally, that $P(Z < -a) = 1 - P(Z < a)$. All agreed, however, that the most important element in explaining either of these was the diagram that was used.



In the diagram it is unimportant whether the point on the x-axis is labelled a or 2. Either could easily be replaced by any other positive number. There is, however, some work to be done in adapting the diagram to deal with the case where a is negative. The domains of validity of the statements represented by the equations $P(Z < -2) = 1 - P(Z < 2)$, $P(Z < -a) = 1 - P(Z < a)$ and the diagram, are a significant area of difference between the representations. In each case the domain of validity is probably left to the reader, and the teacher's intention may well not have included negative values for a , since these require a 'different treatment'. The equation $P(Z < -a) = 1 - P(Z < a)$ however, can stand without adaptation for all real values of a . The equation $P(Z < -2) = 1 - P(Z < 2)$ invites generalisation to other positive values, possibly only to positive whole numbers for some students. The diagram invites generalisation because it is not specific about the value. It is however specific that the value is positive.

In this way the diagram invites generalisation in a different way from the equation. Although it is particular it is easily seen to represent the general, but within the bounds of a certain domain.

The diagram is an imperfect representation of a mental image. As such the same marks on paper can represent different levels of generality for different people. The placement of a 'general point' on a diagram might represent the choice of particular values of the coordinates to one person, or to another it might be a means of representing a situation so that relationships which are true for any such 'general point' can be identified. The difficulty for the student and teacher is that these two understandings give rise initially to the same behaviour.

In addition the language which I used with students to describe the placing of a 'general point' is potentially misleading. In my lessons on these topics in school A I frequently used the language patterns 'it could go anywhere', 'it could be anything' and 'they could be anything'. This was true when I was talking about questions similar to Problem J 'Find, in terms of a and b , the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$ ', where the values of a and b are unconstrained. It was also true when I was talking about locus questions where the literal symbols represented the coordinates of a point on a particular curve and therefore were constrained by a relationship with each other. In these cases my image was of a point which was initially unconstrained, but which became constrained by the relationship as I developed the equation expressing this relationship. The (a, b) which I placed on the diagram was unconstrained. The diagram was to help me express distances in terms of the unconstrained quantities. My formation of an equation then produced the constraint. For the students the position of (a, b) was always constrained by the conditions in the question, hence my insistence that the point could go anywhere was unhelpful. Later in the year, when my focus was on the geometrical aspect of the question, rather than on the expression of a relationship between variables, I too adopted the stance that the position of (a, b) was constrained from the outset.

My language patterns seem to have developed from a response to the question students ask, faced with the task of placing the point on the diagram, 'where should I put it?'. These responses are inadequate to the immediate query and do little to address the insecurity revealed by the question.

Peter's responses to the class's early work on locus questions, along with the responses of two other teacher groups (one Polish and one English) suggest that the language patterns I used in these situations are commonly used by other teachers.

Peter was an enthusiastic and well-informed teacher of six years' experience yet he did not anticipate the nature of the problem that some pupils were having.

Speaking to me the day following the lesson he said that dealing with (a, b) as a general point was so familiar to him that he had not been able to see where the difficulties lay for the pupils. Like Daniel, he was able to use my presence to develop his awareness of his own mathematical awarenesses. He began to appreciate some of his taken-for-granted expertise.

The expression of generality, whether by use of literal symbols or by use of diagrams requires a set of awarenesses which the expert has begun to take for granted. Trivialisation of such awarenesses by the use of careless language patterns may inhibit the students' enculturation into the expert awarenesses.

Seeing the Particular in the General

At the beginning of the last section 'Expressing Generality' I asserted that students of 'A' level mathematics are more often called upon to use a general formula in a particular case, that is to specialise from an expression of generality, than to derive their own expression. Recognising that a particular case can be treated as one example of a generality is however no trivial task. The general expression must be seen to represent a number of particulars and the particular must be recognised as conforming to the general form.

Frank's work on factorising cubics brought to my mind the distinction between deriving a form and seeing the particular represented in it as a general statement. I had asked him to work on Problem E 'Factorise $x^3 - 1$, $x^3 - 8$ and $x^3 - a^3$ ' and he had done each of them, with my help, by using the factor theorem to identify the linear factor and then long division to find the quadratic factor (the transcript extract is from 'Frank', lines 307 to 320). After successful completion of the third I asked

Liz: Right. Okay. Can you say anything about that third one in relation to the first two.

Frank: ... I found it easier. I think that might have just been the practice, I'm not sure. Even though you're dealing like with two variables and it is just the same, just as, exactly the same method - there's no change.

Liz: Okay. Supposing I asked you to do erh $x^3 - 27$. What would you do?

Frank: Umm it would be $x - 3$ and then I'd lay it out just like that and $x^3 + 0x^2 + 0x - 27$ - and then I'd just work through it.

Frank clearly envisaged working through the long division again to obtain the quadratic factor. He had recognised the similarity between the first three cases, saying, 'it is just the same, just as, exactly the same method', and that the fourth case conformed to the pattern, at least to the extent that 27 is 3^3 . But he had not

seen that the general case $x^3 - a^3$ contains all the others within it. Frank saw the factorisation of $x^3 - a^3$ as just another factorisation which happened to have the letter a in it, rather than seeing it as a summary and prediction of particular factorisations.

At a meeting of mathematics teachers in Poland I spoke about my students' work on this cubic factorisation. The response of a number of the teachers was that their students would have been taught $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$ as a general formula and then expected to apply it to particular cases. One commented that the students often found it difficult to apply a general formula, giving as an example

$$120^2 - 119^2$$

which invites the use of the formula $a^2 - b^2 = (a + b)(a - b)$

He said (translated by an interpreter):

'It is, as you see, one of many exercises when students want to use a calculator and do not want to use the general formula like this.

It is easier with them to deal with the general formula. It is very difficult to apply it.'

In each of these cases the 'form', $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$ or $a^2 - b^2 = (a + b)(a - b)$ remains general and empty of content. The form does not 'inform' their view of the mathematical world.

Summary

In this chapter I have discussed various aspects of the relationship between particular and general as evidenced in the work of the sixteen and seventeen year old students I have had contact with. I have established a number of ways of characterising and distinguishing between students' acts of generalisation and explored some ways in which teachers attempt to assist these shifts from particular to general, including the use of generic examples. This involved me in a lengthy discussion of the notion of a generic example, including those features seen as important by me and by other teachers. I have also described how my students responded to new kinds of relationship between particular and general created by the interface between algebra and geometry in a study of coordinate geometry. Teachers' enculturation into the notion of placing a 'general' point on a diagram was seen to blind them to their students' different understanding of the situation. I have also examined two situations in which the general form was not seen by students as a summary of particulars, but merely as another particular case which included letters.

An issue which has arisen from time to time in this chapter is that of the relativity of the terms particular and general. For example in Problem A 'For which values of k is $k(k - 1)x^2 + 2(k + 3)x + 2$ positive for all real values of x ?' my intention was that the students should particularise by taking numerical values for k in order to generate particular functions of x . Some of the students, however, looked at particular values of x and k so that they generated numerical values for the particular functions. The particular functions are particular relative to the original expression $k(k - 1)x^2 + 2(k + 3)x + 2$, but general relative to the numerical values generated by substituting for x .

This example also gives an indication of the ambiguity in the role of k which allows it to be seen as either particular and general, or perhaps more accurately as something between the two.

In the next chapter I will explore ways in which this ambiguity is expressed and understood by students.

Chapter 6 Roles of Literal Symbols

In this chapter I discuss the roles of variables and the literal symbols which represent them. In particular I look at variables in relationship to each other in problems where several variables are involved. A central theme is the complexity of the way in which literal symbols are used to express the particular or general nature of the variable, and students' struggle to deal with ambiguity in the roles of some variables.

In order to do this I discuss a number of mathematical tasks and the ways in which they have been approached by students, by colleagues and by me. My intention is that you, as reader will spend a little time working on each task as you come to it.

Order of Variation

In this section I return to the issue of quantifier structure which was raised in chapter two in my discussion of the work of Bloedy-Vinner. She claims that understanding of algebraic language in the context of parameters is characterised by a correct ordering of the quantifiers. My intention is to show that understanding of the quantifier structure of a problem cannot be inferred from the solution and to introduce another way, which I have called 'order of variation', of distinguishing between solution processes.

Two tasks

I Sequences

You are invited to spend a few moments working on this task:

Problem K Explore sequences generated by the rule $x_{n+1} = ax_n + 1$ for various values of a .

I have given this task to about fifty pupils, teachers and research colleagues in various settings. Most of them have taken one of two broad lines of approach.

The first approach was to choose a value of a and then produce a number of sequences from different starting values (x_0). For example a pupil chose $a = 2$ and subsequently produced the sequences

$x_0 = 0$	0, 1, 3, 7, 15, ...
$x_0 = -2$	-2, -3, -5, -9, -17, ...
$x_0 = 27$	27, 55, 111, 223, 447, ...

Another chose $a = -0.2$ and produced the sequences

$x_0 = 0$	0, 1, 0.8, 0.84, 0.832, 0.8336, 0.83328, ...
$x_0 = 2$	2, 0.6, 0.88, 0.824, 0.8352, 0.83296, ...
$x_0 = -15$	-15, 4, 0.2, 0.96, 0.808, 0.8384, 0.83232, ...

People taking this approach usually made some abstraction about the nature of such families of sequences. For example, with $a = -0.2$ each sequence has a specific common limit, whereas with $a = 2$ all but one of the sequences generated by different starting values are unbounded (the exception is the sequence starting from $x_0 = -1$). Most then moved on to another value of a . Eventually some were able to make general statements about how the nature of the sequences, that is whether they will be bounded or unbounded, can be predicted from the value of a .

The second line of approach began with settling on a value of x_0 and using the same starting point for every value of a . For instance a pair of teachers chose to use 1 as the starting number for each sequence they produced. They then produced sequences such as:

$a = 0$	1, 1, 1, 1,
$a = 1$	1, 2, 3, 4, 5,
$a = 2$	1, 3, 7, 15, 31,
$a = -5$	1, -4, 21, -104, 521

This approach also gave rise to general statements about the nature of the sequences in relation to a , for example that if a is negative the terms in the

sequence alternate in sign. As a second stage some who used this approach went on to look at the effect of changing x_0 .

A variant on either of these approaches was to consider not the final behaviour of the terms in the sequence (i.e. whether the sequence is bounded or unbounded) but the terms themselves, sometimes expressing them as functions of n . This emphasis was made by a colleague who used the graphing facility of a spreadsheet to explore the sequences and by a teacher who began sketching graphs in response to the sequences he was generating on his calculator.

My aim in presenting this task is to draw your attention to the issue of hierarchy of variables. I have asked you to work on the task yourself so that you can link what I have to say about it with your own experience of this and of other tasks. You may connect others tasks with this one metaphorically or metonymically, that is by an experience of resonance ('This is like ... '), or of triggering (some aspect of this task triggers a memory of another task even though they are largely dissimilar).

The two broad approaches I have described use different hierarchies. One holds a constant while varying x_0 and the other holds x_0 constant while varying a . (The variant I have described may be seen as holding both a and x_0 constant and varying n).

As well as representing two approaches to working on the problem by specialising, these two approaches represent two ways of seeing the problem situation:

- seeing each value of a as representing an iterative rule which will generate sequences. Common features of the behaviour of these sequences may then be expressed verbally or symbolically.
- seeing each value of x_0 as labelling a set of sequences. The features of each sequence depend in some generalisable way on the value of a used to generate it.

These ways of seeing are not, of course, mutually exclusive and do not cover every possible approach to the problem.

One further line of approach was taken by a small number of colleagues in response to this task. This was a very general approach, choosing not to specialise for a or x_0 at all. One colleague wrote:

$$\begin{aligned}x_{n+1} &= ax_n + 1 & \text{if } a \geq 1 & \text{divergence} \\x_1 &= ax_0 + 1\end{aligned}$$

$$x_2 = a(ax_0 + 1) + 1 = a^2x_0 + a + 1$$

$$x_3 = a(a^2x_0 + a + 1) + 1 = a^3x_0 + a^2 + a + 1$$

$$x_n = a^n x_0 + a^{n-1} + a^{n-2} + \dots + a + 1 \quad (\text{assume this and use induction})$$

$$\text{limit } \therefore x = \sum a^j = \frac{1}{1-a}$$

Her method involved maintaining complete generality in the values of a and x_0 but specialising to small values of n in order to achieve a generalisation for the value of a term in a sequence defined by x_0 , a and n . I could describe her method as 'varying n ' but without keeping a and x_0 constant at specified values. The succinctness of her written expression makes it difficult to know how she saw the problem situation. Her mental image may have included a collection of individual sequences determined by the values of x_0 and a . Or she may have seen only a general sequence, not exemplified by particulars. Alternatively the symbols may have more forcibly represented expressions on which to operate and the connected ideas of limits, geometric progressions and proof by induction may have been the more salient context.

On a number of occasions I have asked colleagues to work on this problem and afterwards discussed with them the roles of the three variables in the problem (x_0, a, n) and in particular the effect of varying them in different orders. Some were unhappy with the idea that x_0 , a and n had at all comparable roles. They had worked on the problem by the first approach I have outlined, that is by holding a constant in the first place whilst allowing x_0 to vary. It was not until they had seen the method used by another colleague, whose work broadly followed approach two, that they expressed an appreciation of what it might mean to vary a first. One interpretation of this reaction is to say that their mental image of the problem situation was inflexible, so that they could not conceive of addressing the problem by a different route.

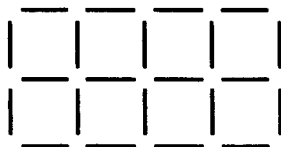
On a separate occasion I had asked a group of teachers to work on the task without initially giving any reason for my interest in it. Two of the teachers made a conscious decision, after a discussion, about what to vary first. During this discussion varying n first was raised as a possibility. A third teacher was very dismissive of this possibility. In the discussion following the work on the question, however, a variety of approaches was identified and acknowledged by the group. Working on and discussing the question together forced a widening of views on acceptable solution procedures and the ways in which they might be described. It widened awareness of the different perspectives that might be taken on this problem and the kind of experiences and conclusions available through each.

Each broad approach represents a way of seeing the problem situation and involves a set of awarenesses. Having only one or two of these sets of awarenesses activated limits the kind of responses I can make as a learner in solving the problem and reduces my ability as a teacher to respond to the work of others.

II Rows and Columns

In chapter five I described my work on the following problem:

Problem C The picture below shows a rectangle made up of two rows of four columns and of squares outlined by matches. How many matches would be needed to make a rectangle with R rows and C columns?



When I worked on this problem I began by producing some diagrams with two rows of squares and tabulating the number of matches against the number of columns. From my table I produced a formula $M = 5C + 2$.

Next I changed the value of R to 3 and, with the aid of one or two diagrams, convinced myself that M and C now fitted the rule $M = 7C + 3$.

Similarly, I found that, for $R = 3$, $M = 7C + 3$, for $R = 4$, $M = 9C + 4$ and, for $R = 5$, $M = 11C + 5$.

Now I saw a pattern amongst these rules which allowed me to formulate the more general rule $M = (2R + 1)C + R$.

My reason for drawing this problem and solution process to your attention once more is to point out that my decision to vary C whilst holding R constant rather than varying R whilst holding C constant was (mathematically) arbitrary. The alternative choice would have made no difference to my working or the kind of conclusion that would be available to me. Mathematically the roles of the two variables are symmetrical.

This is in contrast to the 'Sequences' task in which the different orders of variation were representative of different images of the task situation. A decision to hold x_0 constant while varying a gave rise to a different solution method and a different set of potential conclusions from those available following a decision to hold a constant while varying x_0 .

Quantifier Structure

I described briefly in chapter 2 how the notion of 'order of quantifiers' is explored in Bloedy-Vinner H (1994). She uses this and the term 'dynamics of substitution' to describe what she sees as characterising algebraic thinking in the context of parameters. The example which she gives to explain these terms concerns the question

Problem L In the following equation x is an unknown and m is a parameter:
 $m(x - 5) = m + 2x$. For what value of the parameter m will the equation have no solution?

The quantifier structure for this example is given by Bloedy-Vinner as follows:

- (a1) for all m , there exists $E(x)$ so that $E(x)$ is the equation $m(x - 5) = m + 2x$.
- (a2) for all x , $E(x)$ is either true or false
- (a3) there exists m so that $E(x)$ has no solution
- (a4) there does not exist x so that $E(x)$ holds.

In other words, first m must be fixed to give a particular equation. Then it is possible to decide for each x whether or not the equation is satisfied. Then we must allow m to vary so that we can find the value of m for which no x satisfies the equation. The 'dynamics of substitution' for this question are 'first substitute for the parameter, get an equation, then substitute for the unknowns or variables to check if the equality holds' (p90).

Of course, the quantifier structure and dynamics of substitution are not necessarily paralleled in the solution process. In particular, there is no need to substitute particular values in order to solve the problem. My working might be as follows:

$$m(x - 5) = m + 2x$$

$$mx - 5m = m + 2x$$

$$mx - 2x = m + 5m$$

$$x(m - 2) = 6m$$

$$x = \frac{6m}{m - 2}$$

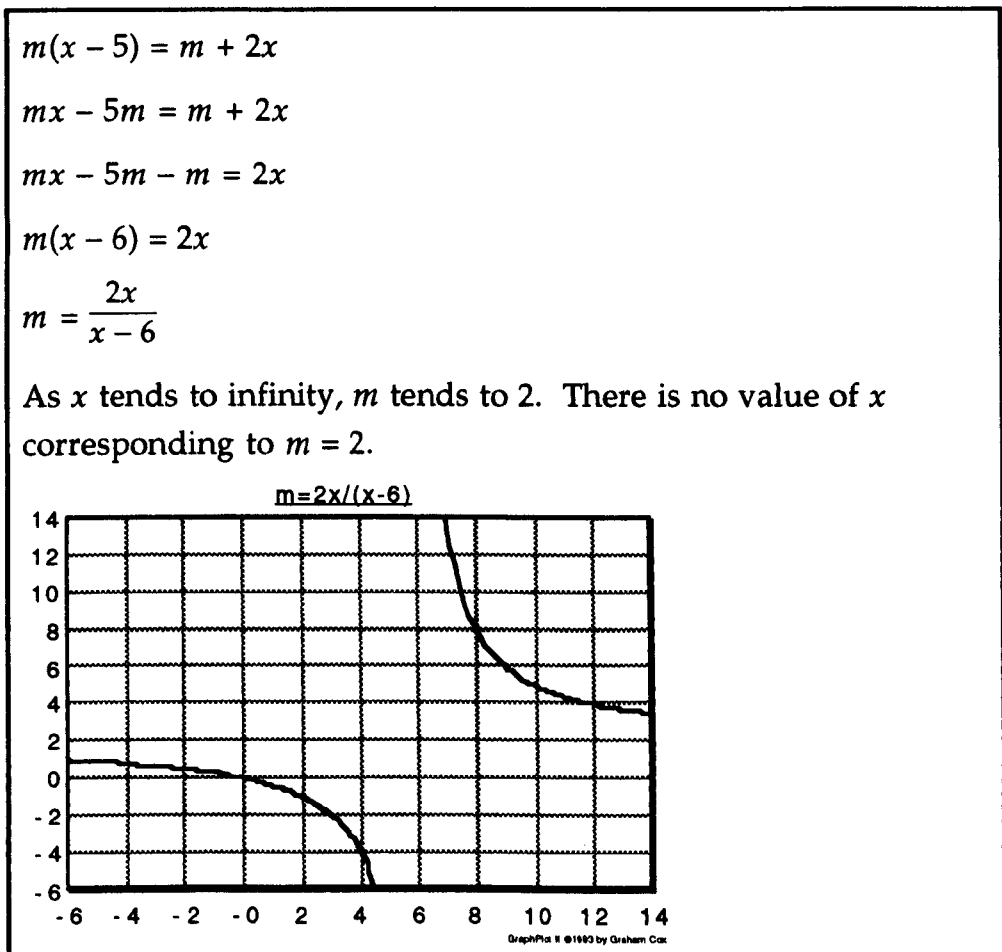
$\therefore x$ is not defined if $m = 2$

For the first part of this working I have seen x as the unknown and the equation as an equation *in* x . The role of m is to represent any number and to remain, until after the equation is solved for x , unquantified. That is, I mentally hold m fixed but without assigning it a particular value. Then in the

final line my attention turns to the value of m . A condition (that the expression $\frac{6m}{m-2}$ be undefined) must be satisfied by m , and so m becomes the unknown. The 'order of variation' in my solution, then, is first x , then m . My solution acknowledges the quantifier structure suggested by Bloody-Vinner although it does not follow it.

My analysis differs from Bloody-Vinner's in that I have analysed one possible solution procedure and its order of variation, whereas she has analysed the quantifier structure of the problem. My contention is that the order of variation is a feature of an individual's understanding of a problem and not of the problem itself. The quantifier structure of the problem does not necessarily correspond with the solution process and it is not possible to infer an individual's understanding of the quantifier structure from their method of solution.

As we have seen, different orders of variation are possible in the 'Sequences' and 'Rows and Columns' tasks. Here is another possible solution to Problem L



In this solution I hold x constant but unquantified and treat m as an unknown in the first stage. Secondly I consider varying x , and the value(s) which my expression for m could not possibly take as x varies. The order of variation is

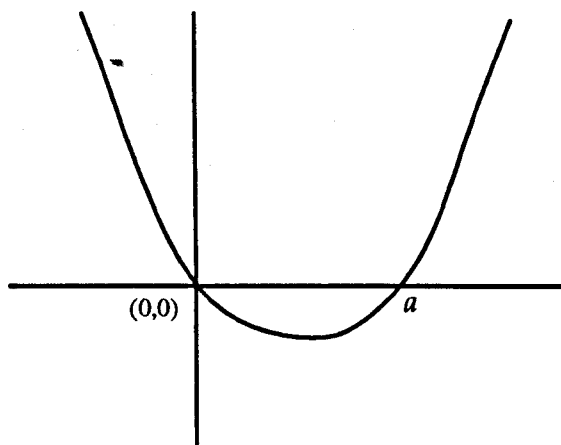
the reverse of that in my previous solution although the quantifier structure which underlies the solution is unchanged.

Students Talking

An Unexpected Ordering

An individual student, Eddie, was working in my presence on a task I had set him (the transcript extract is from 'Eddie', lines 14 to 23). The task was

Problem M Sketch $y = x(x - a)$



He made this sketch of the curve without my assistance by considering the general shape of the graph (he multiplied out the bracket in order to decide that the equation was quadratic and the curve therefore a parabola) and its orientation. He also identified the points at which it cut the axes.

Liz Marvellous. Tell me about the role of a in that equation.

Eddie Er, well the a would be the coefficient of the x . So, whatever value x is .. it would be $x^2 - xa$ so if a is greater than x it would be a negative number

Liz Right

Eddie Or well it depends if they're negative or not, but if a is less than x then it would generally be a positive number

Liz Uhmhm. What would happen if you vary what a is then?

Eddie Er, as you vary a if a is a low number - like take x is a constant, say 1, for example,

Liz Yeah

Eddie So you'd have 1 squared, which is 1, minus 1 times a which is a ... as a rises y becomes a lower number taking x as 1 .. and as x is lower y becomes higher so there's an inverse relationship between y and a in this case.

Liz Right that's when x is 1

Eddie That's when x is equal to 1, yeah.

My reaction to his words was one of surprise. I have also noted surprise as one of the visible initial reactions amongst my teacher colleagues when I have recounted this incident. I expected Eddie to describe the effect on the graph of varying a . In other words I expected him to treat the graph as an entity, as a summary of the relationship between x and y . I expected him to demonstrate that he had 'encapsulated' (Dubinsky 1991) the process of calculating y from x in the graph as an object. However, faced with the complexity of an equation linking three variables, he simplified by holding one of the three variables constant, rather than by summarising the covariation of two of the variables as an image of a graph. I expected Eddie to hold a constant in drawing the graph and then to allow a to vary and produce different graphs.

To see a as a parameter giving rise to a family of curves is to acknowledge a certain 'order of variation'. I must first hold a constant and allow x and y to vary. This allows me to form an image of a curve. Then I may allow a to vary so that a family of curves is generated. Eddie's approach, in answering my question about the role of a , was rather to hold x constant while exploring the relationship between the variations in a and y . We made little progress in our conversation because my attention was fixed on one order of variation and his on another.

My reaction of surprise is an example of an incident becoming salient because of a lack of fit between my own awarenesses (in this case mathematical awarenesses) and the student's apparent awarenesses. My expectations regarding the order of variation in this problem were so strong that they precluded my being prepared for Eddie's answers.

Role of the Graphic Calculator

Another student, Lorne, worked on the same problem (the transcript extract is from 'Lorne', lines 134 to 148):

Lorne: Umm $y = x(x - a)$. Umm, well, first of all, expand it – is that the correct term for it?

Liz: Yes.

Lorne: x^2 and umm $-xa$. $y =$, is that right, is that right? Umm that would probably be, I don't know I probably need a table or, perhaps. .. yes may as well. No. Umm

Liz: Think of some curve sketching techniques.

Lorne: Well it's going to be a .. parabola.

Liz: Uhmhm.

Lorne: It's going to, oh umm, I need my graphical calculator. I wouldn't, wouldn't do this, I wouldn't actually do this if I didn't have my graphics calculator. It's such an invaluable type of thing. I definitely need it.

Liz: Have you got it with you?

Lorne: Yes, I have.

Liz: Okay.

Lorne: (*he gets out his calculator*) I mean, what I'll probably do is, is a trial and error sort of phase but it won't be like an error. What I'll probably do is $a = 1$ and then 2, 3, so if this one's 1, then it's umm graph $x^2 - x$... right .. goes slightly, goes through the origin. Okay. $x^2 - x$ which goes through origin and (0, 1). $a = 2$ slightly move it down and go to 2 I think. umm, $2x$ yes it goes through 2 sorry. I guess 3 ... will go through 3, origin and 3, (0, 3). So whatever a is it goes through, which means that .. if I was to generalise it, I'd draw a parabola which crosses at the origin and climbs back up at point (0, a).

Lorne was not so well practised as Eddie in curve-sketching techniques. Later in the same conversation I quizzed him about finding the points of intersection between a curve and the axes and his recall of these techniques apparently failed.

Instead Lorne called on his graphic calculator to produce the images of individual curves. The variation of x and y which is implicit in drawing curves from the equations $y = x^2 - x$ and $y = x^2 - 2x$ was performed by the calculator, whilst Lorne as user varied a . The order of variation expected by me as teacher was imposed by the technology in use.

Because Lorne had generated the individual graphs by using the calculator he had avoided explicitly varying the values of x and y at all, whereas Eddie made substitutions of 0 for each of x and y in order to identify the points of intersection with the axes. For Lorne, the features of the graph which he uses

in picturing the general $y = x(x - a)$, that is its passing through the origin and $(0, a)$, were features of its appearance only, and were not connected to the form of the equation. There was no link between particular points on the graph and the substitution of particular values of x or y into the equation. For Eddie these connections existed but the one-to-one correspondence between graphs and values of a was not established.

Summary

One way of distinguishing between the perceived roles of variables in problem situations is to consider the order in which they are varied. This order is a feature of the solver's view of the problem rather than of the problem itself and the solver's perception of this order will influence the construction of a solution procedure. Orderings which are different from the conventional one may nevertheless lead to a solution to the problem. The teacher's awareness of these possible orderings and the images of the problem situation which give rise to each will be preparation to respond to students' solutions.

These orders of variation are an indication of the learner's perceptions of the roles of the literal symbols in their representation of particular and general. The possibility of different orders is one aspect of the complexity of these roles in second variable situations.

Stereotyped Roles of Literal Symbols

Spend a few moments working on this problem:

Problem N Find the equation of a straight line which passes through the point (m, c)

I asked a group of my colleagues to work on this task for me. Most of them had been schooled in England and had been used to seeing m and c in the context of the equation $y = mx + c$, which is almost universally used here as a 'general equation of a straight line'.

One of my colleagues, Naomi, arrived a little late, after everyone else had begun to work and I just passed her a piece of paper with the task written on it as above. After a few minutes for working I began as follows:

Liz: What the question was about as far as I was concerned was that m and c very often turn up in the context of straight lines - in most English schools the form $y = mx + c$ is used as the standard form for the equation of a straight line - but they don't turn up in the way in which I've set them into this question. m and c are now the coordinates of a point that the line goes through, rather than the gradient and the ..

Naomi: (*in surprised tones*) Oh!

Liz:.. intercept

Naomi: (*laughs*) Do I need to make any more comment?

Having arrived in a hurry and with her mind full of the meeting she had just come from, she had glanced quickly at the request on the paper and had simply written as her answer $y = mx + c$. She commented later 'I wondered why everybody else was working so hard!'. Her familiarity with the usual roles of m and c in the context of straight lines had caused her not to see the non-standard meaning I had given them in this task.

Other colleagues acknowledged a discomfort at seeing m and c in the 'wrong' roles and went about dealing with this discomfort in a variety of ways.

One, Frank, recognised that there were 'letters used in the wrong way' but found that the recognition 'didn't actually make it any easier'. Use of m and c as gradient and intercept in the context of a straight line was so ingrained that recognition of the 'wrong use' of these letters was not enough to entirely remove the difficulty and discomfort. He used the form $\frac{y - y_1}{x - x_1} = k$ for the equation of a straight line through (x_1, y_1) with gradient k . He described this as having used a 'translation', since the formula that he carried in his head had m in the role of gradient. He had to maintain a conscious effort to use k in a role which he thought of as m .

A second colleague, Edward, said

'I wrote down $y = mx + c$ despite having already seen that m and c were taken, and so I thought "well I can't have m and c - they're taken, but m and c are the letters that hold the template slots for me so I'm going to call them m' and c' to keep the link but to have the separation". And then I substituted in the value at the point and got an equation, remembering that it is m' and c' that I actually want to keep an eye on. So m and c look like they're parameters but actually they're numbers.'

Edward also 'translated' the form in which he held his formula, $y = mx + c$, into $y = m'x + c'$ in order to avoid using m and c in the usual roles. However he chose to use symbols closely related to m and c because he needed to 'keep the link'. He then had to attend to the difference between m and c and m' and c' . He describes this as 'keeping an eye on' m' and c' . These two are now the quantities for which he needs to find an expression. They have taken on the role of 'parameters' in Edward's language. By contrast m and c are not to be made the subject of an expression but are to be taken as given. The change of role of m and c needed constant attention throughout the solution process.

A third colleague, Deborah, used the form $y = \text{gradient} \times x + \text{constant}$ in place of $y = mx + c$. She said

'And I actually decided that the way I was going to hang on to it was to have "gradient" and "constant" written out as long as I needed to.'

Deborah chose to 'translate' her formula by using words as placeholders rather than the familiar m and c . She perceived this as a way of 'hanging on to' her familiar form and spoke of using this 'as long as I needed to'.

These three approaches have several common features. One is a decision to adapt the familiar version of $y = mx + c$ or $\frac{y - y_1}{x - x_1} = m$ by replacing m and c by something else. The replacement was, variously, by a completely different letter, k , by two closely related symbols, m' and c' , and by words, 'gradient' and 'constant'.

This replacement is referred to by Frank as 'translation', as though the relationship between m and the quantity it represents is like that between signifier and signified. Using k instead of m is like trying to express my thoughts in an imperfectly mastered second language, rather than my mother tongue. An extra stage, of re-interpreting from the familiar language into the second language, becomes necessary. My lack of fluency with this second language means that this translation must be made word-by-word, non-idiomatically. The replacement is at the level of symbol for symbol, rather than equation for equation or algorithm for algorithm. My thinking is done in terms of the familiar symbols and then these are replaced by my new symbols. The old symbols were tools for thinking, that is they were the vehicles for my thoughts but the new symbols cannot fulfil this role.

A second common feature is a feeling of insecurity, discomfort or strangeness. Frank reported 'I was bearing in mind all the time that m and c were something else' and 'still very conscious all the time I was doing it'. Edward had to 'remember that it is m' and c' that I actually want to keep an eye on'. Deborah

had to 'hang on to it'. Again I can draw a parallel with speaking in a second language. There is a loss of fluency. The solution procedure becomes stilted and uncertain, with processes which would otherwise be automatic requiring attention.

Another common feature is the use of an internal monitor. Each of these three was able to divide their attention between performing the mathematical task and observing their mental actions in so doing. They were encouraged to do so by the fact that the task was in a sense familiar, and so needed little of their attention, and in a sense unfamiliar, forcing their attention on to the way in which they dealt with this unfamiliarity.

A fourth colleague, Ian, did not write down a 'translation' of a template as had each of the other three. However he used a mental equivalent of $y = m(x - x_1) + y_1$ as a template for the equation of a line with gradient m passing through (x_1, y_1) . Having written $y = n(x - m) + c$, he was 'very put out that I then had to put a $+ c$ and not have it of the form $y = mx + c$ '. His template involved a 'translation' of m into n but did not contain c as the intercept. The ambiguity of meaning of c was then not in the foreground in the way that the ambiguity of m was. It had not been 'dealt with' in the translation. Finding $+ c$ *not* in the role of intercept was disturbing even though it had been used in this different role.

The use of the symbols m and c in this context was well established and ingrained. In fact, during the subsequent discussion Edward began using c as a euphemism for 'intercept' without realising it, even though our conversation, which had been about the 'other' roles of m and c in this problem, must have served to heighten awareness of this unthinking use. For each of these colleagues the requirement to use these letters in an unfamiliar role in a familiar situation was an off-balancing experience.

Another aspect of this problem which excited discussion was the indeterminate nature of the solution. My question asked for the equation of *a* straight line. Some saw this as asking for the equation of a general straight line through (m, c) , whereas others thought of finding a particular straight line. In this context the terms particular and general need to be used cautiously, since any response to my task involving m and c will necessarily be general, in that these letters refer to any point of the plane. However the distinction can be made between an equation which describes a particular straight line once m and c have each been given a numerical value, and one which still remains general because its gradient is not determined.

The response to this aspect of the question varied. Of those who thought they were being asked for the equation of a particular straight line (in the sense that it would be determined once values had been given for m and c), many found a general equation first and derived a particular equation from it. This conforms with Krutetskii's (Krutetskii 1976) observations of mathematically capable students, who solved problems in general first and specialised the solutions in a particular case afterwards.

The equation of a line which remains general (an example in this case might be $y = k(x - m) + c$) can be thought of as describing, in one equation, a family of lines. Alternatively I can see it as a single line with an unknown gradient, or again as a single line with gradient k . The difference between these is small but psychologically significant. The difference between using a letter to stand for an unknown particular and using it to stand for the general, as in the case of k here, is amplified by the account below, 'Particular becomes General'.

The indeterminate nature of the solution also raises the issue of the agent of choice, that is the identity of the chooser. My colleague who produced the answer $y = k(x - m) + c$ to my question had been obliged to use the letters m and c in his equation, whereas k was his choice. Once the equation is formed it may appear to an outsider that the roles of m , c and k are similar. Each represents a general quantity or each represents a particular but unknown quantity. But to the initiator the experience of putting k into the equation may be entirely different from that of putting m and c into it. m and c are chosen by the question setter and may be treated, by an expert, in just the same way as if they had been 3 and 2. k , on the other hand, has been consciously selected. It therefore must be seen to be significantly different from a numerical gradient.

I see parallels between this choice of k as gradient and the placing of a general point on a diagram, as described in chapter five. In each case the experienced user understands the nature of the freedom which they have in making the choice, whereas to the novice the constraints on his choice are still confusing.

As an expert I am aware of my ability to choose a letter when some expression of generality is required. Students may be much more restricted in their awareness of this choice. One of the factors which restricts their awareness of choice is the conventional use of certain letters in certain roles. For example, in answering 'Find the general equation of a line which passes through the point (3, 2)' students may use the letter m to stand for the gradient without experiencing any choice. I will describe choices dictated by common usage in this way as culturally determined. In answer to the question 'Find the equation of a line with gradient m which passes through the point (3, 2)' the choice of m

is mathematically necessary. For students these two questions and the choices implied may be indistinguishable.

Particular becomes General

As an 'A' level student I remember working on many questions similar to this one:

Problem O The normal to the curve $y^2 = 4ax$ at the point $P(ap^2, 2ap)$ meets the curve again at Q. Find the locus of the mid-point of PQ

My working on this question would have been something similar to the following:

For $y^2 = 4ax$,

$$y = 2at, x = at^2 \quad \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

The gradient of the normal at P is given by $-p$, so the equation of the normal at P is

$$y - 2ap = -p(x - ap^2)$$

$$y + px = ap^3 + 2ap$$

The normal intersects the curve at $Q(aq^2, 2aq)$, say. So Q lies on the normal and

$$2aq + apq^2 = ap^3 + 2ap$$

$$ap(p^2 - q^2) + 2a(p - q) = 0$$

$$a(p - q)[p(p + q) + 2] = 0$$

Since $p \neq q$,

$$p(p + q) + 2 = 0$$

$$p + q = -\frac{2}{p} \text{ and } pq = -2 - p^2$$

The mid-point of PQ has coordinates $(\frac{a(p^2 + q^2)}{2}, a(p + q))$

$$\text{But } p^2 + q^2 = (p + q)^2 - 2pq = \left(-\frac{2}{p}\right)^2 - 2(-2 - p^2) = \frac{4}{p^2} + 4 + 2p^2$$

$$\text{And } p + q = -\frac{2}{p}$$

So the mid-point of PQ has coordinates $(\frac{a}{2}(\frac{4}{p^2} + 4 + 2p^2), -\frac{2a}{p})$

The coordinates of the mid-point of PQ satisfy

$$x = \frac{a}{2}(\frac{4}{p^2} + 4 + 2p^2), \quad y = -\frac{2a}{p}$$

Eliminating p gives

$$x = \frac{y^2}{2a} + 2a + \frac{4a^3}{y^2}$$

which is the locus of the mid-point.

The reason why these problems are especially memorable is that there was a moment in their solution which I found especially pleasurable. That moment came right at the end of the solution when a pair of coordinates for the mid-point of PQ was transformed into the equation of a locus. Until then everything about the problem had, in my mind, been particular. I had treated the curve $y^2 = 4ax$ as a particular parabola, P as a particular point on the curve and Q as the corresponding point on the curve, whose position was determined by the position of P. The mid-point of PQ was then also a particular point. Even though the use of a and p gave the situation a wider generality, in my mind P was a particular point on a particular curve. I worked as though a and p took particular values. In order to make the last stage meaningful all this had to change. P had to be allowed to move on the curve in order to generate the locus. From being a particular value of the parameter t , p had to become the parameter itself, so that as it varied the whole curve $y^2 = 4ax$ was mapped out. The power which resulted from this ambiguity in the meaning of P was the source of the moment of pleasure.

Another reason why I remember so clearly my experiences of working on these problems is that at first the final stage of the working was obscure to me. Over time I gained an appreciation of the meaning of that stage and finally found pleasure in it.

This account is from the perspective of an adult reflecting with adult mathematical and other awarenesses on the experiences of a teenager. As a teenager I noticed that I had not previously understood what I was doing in these solutions and now I gained pleasure from them. As an adult I account for this pleasure by engaging my adult awarenesses of the shifts of attention required by the problem situation. The shift is from seeing P as a particular point to allowing it to vary and hence generate another curve.

Bound up with allowing p to vary and generate another curve is the naming of the coordinates of the mid-point of PQ as x and y . The particular meanings of x and y in the context of coordinate geometry are as variables expressing a relationship which gives rise to a curve. So the labelling of the coordinates of the mid-point of PQ as x and y was part of an acknowledgement that this mid-point was a *general* point on a curve.

Colleagues talking about this kind of problem (for instance John below) have spoken of 'inventing' other names for these coordinates, for example h and k or X and Y . There is a sense in which x and y are already 'taken' and different names must be found. The reluctance to use x and y in these circumstances perhaps reflects the different status of the curve $y^2 = 4ax$, the normal $y + px = ap^3 + 2ap$ and the 'new' curve. The 'new' curve is not yet seen as a curve but as a relationship between the two coordinates. The alternative names would then be replaced by x and y in the final stage of giving the equation of the locus. So acknowledgement of the mid-point as a general point on a curve still takes place and is still accompanied by a naming of this point as (x, y) .

Shifts provoked by the need for a different letter

This aspect of my analysis of these A-level problems alerted me to look for other examples of shifts of attention triggered by the need to use the same letter or letters with more than one meaning. Two examples follow.

Dummy variables

In working on continuous probability distributions it is usual to use the cumulative distribution function $F(x)$ given by

$$F(x) = P(X \leq x)$$

and the probability density function $f(x)$ which is defined so that

$$P(a \leq X \leq b) = \int_a^b f(x) \, dx.$$

The relationship between these two functions may be expressed as

$$F(x) = \int_{-\infty}^x f(x) \, dx$$

or $F(x) = \int_{-\infty}^x f(t) \, dt$ (or the equivalent with any other letter standing for the argument of f)

Teachers may avoid using the first version because it uses x in two different ways. The second may be rejected by students because they cannot accept $f(t)dt$ as equivalent to $f(x)dx$. Such an acceptance requires acknowledgement of the arbitrary choice of x . This acknowledgement may be difficult for two reasons. First, students' experiences of expressions for functions will have involved x as the independent variable, almost without exception. So the use of any other letter in this capacity is seen as strange, and something which requires

exceptional justification, for example that the function is displacement in terms of time and therefore expressed as $s(t)$. There is no such justification here.

Secondly, the acknowledgement requires seeing the definite integral as a function of the limits of integration, rather than as a function of the argument of the integrand. This is in sharp contrast to the indefinite integral. The latter is usually taught first to sixth form students, and the definite integral seen as merely an extension of the process. Since the substitution of limits into the indefinite integral is of a lower order of technical difficulty than finding the indefinite integral, this stage of the process is often under-emphasised. Students, whose lasting impressions of integration are formed during their work on practice examples, come away with the idea that there is not very much difference between the two. This impression is confirmed by the very similar notations used for the two kinds of integral. It is then difficult to see that the choice of argument is entirely arbitrary.

Discriminants

A class of undergraduate students taking a course in mathematics education were working on envelopes of families of straight lines. I had described one set of straight lines as $y = mx + m^2$. Our procedure involved writing the quadratic equation

$$mx + m^2 = ax^2 + bx + c,$$

$$\text{or its equivalent } ax^2 + (b - m)x + (c - m^2) = 0$$

whose solutions are the x -coordinates of the point of intersection between $y = mx + m^2$ and $y = ax^2 + bx + c$. The next stage was to find values of a , b and c for which this equation will always (for every value of m) have equal roots. These values of a , b and c will then identify the envelope of this set of lines. Identifying the values of a , b and c meant finding the discriminant of this equation and equating it to zero. In my experience and that of the students, the discriminant had always been represented by $b^2 - 4ac$. I found the students then writing the following:

$$a = a, b = b - m, c = c - m^2.$$

They went on to substitute these expressions into their 'form' $b^2 - 4ac$ for the discriminant without any apparent problems over the ambiguity of the symbols. After the writing of the three equations above, the symbols a , b and c did not appear in the same equation with two different meanings. So the students needed only to separate mentally the two meanings of each of a , b and c whilst they made the substitution into the form for the discriminant.

When preparing this task to set the students I originally described the family of lines by the equation $y = ax + a^2$. The ambiguity of symbols produced by this choice was of a different order. With this choice of equation, when the equation equivalent to 'discriminant = 0' is formed, there are appearances of a meaning 'coefficient of x^2 in the equation of the envelope' and meaning 'parameter specifying the family of straight lines', both in the same equation. I decided to avoid the confusions possibly arising from this choice.

These two examples point to the difficulty of avoiding multiplicity of meaning whilst maintaining conventional usages. The dilemma in choice of letters created by this difficulty provides an opportunity to explore the nature of the necessity in this choice, that is to point to the distinction between choice dictated by common usage and choice dictated by the need for mathematical consistency. These are the two kinds of dictation of choice which I have labelled as cultural determination and mathematical necessity.

Three examples of stereotypes in action

Helpful stereotypes?

During one of my meetings with teachers David reported:

'They were differentiating and suddenly one of the examples, instead of being to do with θ or x , was e^{-kt} . And I looked at it and I thought "how many of them are going to ask 'which do I differentiate it with - t or k ?'". But they've obviously come across enough examples, very few but enough, to go for the t and not the k - or they've come across k being a constant enough. So they didn't ask, which was a surprise to me.'

Unhelpful stereotypes?

At the summer school for the Open University course M101 'Mathematics: A Foundation Course' one of the standard lectures is concerned with a mathematical model of the rainbow. The model involves the expression

$$A(h) = 2\sin^{-1}h - 2(N + 1)\sin^{-1}\left(\frac{h}{k}\right) - N$$

where h is the height above the horizontal diameter at which a light ray strikes a water droplet modelled as a sphere and k is the index of refraction. The expression for A gives the total angle of deviation between entering and leaving the droplet. The next stage in the solution of the mathematical problem is to find when A takes its maximum value as h varies. The lecturer explains that this will be achieved by differentiating the expression. There are expressions of unease as the audience contemplates differentiation with respect to h .

Pleasure from the unexpected

When I was studying mathematics in school at age sixteen I was asked to solve a problem concerning the maximum area of a rectangle of perimeter 20. I represented one side of the rectangle as x and set up an equation for the area, $A = x(10 - x)$. I can still remember the pleasure in realising that I could find the maximum value of this expression by differentiation and that I could use the label $\frac{dA}{dx}$. Until then I had never seen the notation $\frac{d}{dx}$ in any context other than $\frac{dy}{dx}$.

Each of these incidents has in common that it involves a stereotyped expectation of the role played by certain letters. In the first the role of independent variable is expected of t but not of k . In the second the role of independent variable is not expected of h and in the third the role of dependent variable, that is role of naming a function, is not expected of A .

In the first case the stereotyped expectations of the roles of k and t allow the students to perform the task without having to put their attention on the question of which literal symbol they should treat as representing the variable. The expectation allows fluency.

In the second and third cases, the unusual use of literal symbols in this context represents a departure from the accustomed context for the use of the technique of differentiation. The students' prior experience of finding maxima and minima has been in the context of y as a function of x . The reason for the use of a different letter is that the new letter 'stands for' some quantity in the contextual problem i.e. h for height and A for area.

In these two cases students' expectations prevent them from dealing with the non-standard. These examples point to a tension between the development of fluency and the flexibility to deal with unexpected uses of letters. I develop this theme later in the chapter.

What is special about x and y ?

Lorne

Lorne selected this question to work on

Problem P The point (a, b) is equidistant from the x -axis and the point $(1, 2)$. Find an equation linking a and b .

He worked through it unaided by me except that I corrected one or two errors in algebraic manipulation as they arose. He used x and y throughout to stand for

the coordinates of the point referred to as (a, b) in the question (the transcript extract is from 'Lorne' lines 32 to 78).

Lorne: Then $y = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{5}{4}$. (*writes this*) .. That's the answer to it (*writes ANS by this last equation*).

Liz: Okay. Umm .. what was this question about then?

Lorne: Umm the equal distance in, the equal distance between the one point and the other point - locus.

Liz: Right. You said this was like a question that we've done. (*We had done two questions involving equations of parabolae the previous day. One was Problem I, almost exactly similar to this one but involving the point (3, 2). The other was Problem J which asked for an equation in terms of x and y and had directrix $x = -2$, so that the equation gave x as a quadratic function of y.*)

Lorne: Uhmhm.

Liz: What's erh similar and what's different about it? (*I want to draw attention to the type of conditions that give rise to a parabola and to the fact that having a line parallel to the x-axis as directrix as opposed to one parallel to the y-axis gives rise to a parabola with a different orientation*)

Lorne: Umm well it's the same, what I'm doing here is working out the equation of a, of the actual line, but the question says find an equation linking well linking a and b .

Liz: Uhmhm.

Lorne: Which is the same thing isn't it? Or is it?

Liz: You tell me.

Lorne: Umm .., yes.

Liz: So you haven't strictly speaking answered their question, have you?

Lorne: No, not quite, just an equation.

Liz: So if I was being umm pedantic and saying can I have an answer to the question please

Lorne: then umm that's not the answer (*he scribbles out the ANS which he had written next to his equation*).

Liz: (*laughter*)

Lorne: It's, it's

Liz: It was dangerous to write 'answer' next to something.

Lorne: Yes. It's similar to what we've done but what you've got to do is find an equation linking a and b , and umm linking a and b , I mean, I don't know actually umm .. an equation linking a and b . I, I don't, I don't quite understand what finding the equation linking a and b really means.

Liz: Uhmhm. Well what erh what part does a and b play in this question?

Lorne: It means it's any point on the parabola which is this same length between the point (1, 2) and the x -axis.

Liz: Right so it's any point on that parabola that you've sketched.

Lorne: Yes.

Liz: Umm when you wrote this equation down you were referring to a point on the parabola.

Lorne: Yes.

Liz: What did you call it?

Lorne: .. I called it, what do you mean what did I call, I mean I

Liz: Well you were talking about this point here weren't you?

Lorne: Yes.

Liz: What are the coordinates of that point?

Lorne: The coordinates of that point is umm (a, b) , (y, x) umm (p, q) .

Liz: Yes, quite. (a, b) or (x, y) .

Lorne: Anything yes.

Liz: What, you used x and y .

Lorne: Ah ha. So what I could do is put erh, y is b , so $b = \frac{1}{4}a^2$ could I?

Liz: b equals a quarter what?

Lorne: $\frac{1}{4}a^2$ +. So, I'm not, what I'm answering is the equation linking x and y instead of a and b . So it's $b = \frac{1}{4}a^2 - \frac{1}{2}a + \frac{5}{4}$. (*writes this*)
Which is the answer. (*writes ANS by his last equation*)

In his working on this question Lorne used x and y to stand for the coordinates of a general point on the parabola almost unconsciously. His answer to my first question on this issue ('.. I called it, what do you mean what did I call') suggests that he had not recognised my description of his choice of letters. My question was based on the perception that choosing letters to stand for the coordinates of the general point is equivalent to naming that point. A later answer ('The coordinates of that point is umm (a, b) , (y, x) umm (p, q) ') suggests that he had not recognised that he had made any choice. He did not see that his writing of the first equation $y = \sqrt{(1-x)^2 + (2-y)^2}$ implicitly made such a choice. His familiarity with x and y in this role made the choice automatic, that is un-noticed. In fact his utterance, 'what I'm doing here is working out the equation of a, of the actual line, but the question says find an equation linking well linking a and b ', suggests that, in his eyes, for an equation to represent a curve (the actual line) it must be expressed in terms of x and y .

Test question

My students in school A were set a test in January 1994. Among the questions was the following:

Problem Q A circle has centre $(2, 4)$ and passes through the point $(-1, 5)$. The point (p, q) lies on the tangent which touches the circle at $(-1, 5)$. Find an equation linking p and q . Hence write down the equation of the tangent.

Of the students who made any substantial attempt at the question, all but one worked with x and y rather than p and q . Some obtained an equation in terms of x and y and then substituted p and q into it. Some did not include p and q in their answer at all.

The students' attention, I suggest, was on finding the equation of the tangent and the steps on the way to that aim (finding the gradient of the radius and hence the tangent, obtaining the equation of a straight line with this gradient and passing through $(-1, 5)$). In order to focus on these steps they lost sight of the specific detail of this question and its reference to the point (p, q) . They used the letters they were familiar with using for a general point, that is x and y .

These students, along with Lorne, demonstrate that their use of x and y as coordinates of a general point on a curve is almost unconscious. Mention of other letters in the text of a question was insufficient to bring the issue to the surface. When Lorne was challenged about his choice of letters, his recognition of what he needed to do to satisfy the needs of the question was not immediate. His use of x and y was so automatic that it took some discussion before he noticed it.

What is a ?

In January 1994 I recorded the following about a lesson in school A. My conversation during this lesson with Frank and Trevor has already been discussed in chapter five.

After some work on the remainder and factor theorems and on division of polynomials, Peter asks the class

Problem E Factorise

(1) $x^3 - 1$ (2) $x^3 - 8$ (3) $x^3 - a^3$

We both walk around the class looking at students' work. I go to the table where Tommy is working. He is about to begin (3). He asks for my help, saying 'I don't know what a is'.

In this lesson, which was focused on methods of factorising polynomials and solving polynomial equations, Peter wanted to take the opportunity to introduce the students to the factorisations of the difference between two cubes and the sum of two cubes, commonly expressed as $a^3 - b^3$ and $a^3 + b^3$. All the polynomials they had factorised so far were in x and had numerical coefficients. Tommy's statement ('I don't know what a is') betrays that the unfamiliarity of this situation had thrown him back into the state of wanting to evaluate the letter. Notice that he was not concerned that he did not know what x was. The role of x as a variable, that is as a quantity which can take any value and takes no particular value, was well-established. It was the social practice in this school as in many others to write expressions in one variable in terms of x . In this task (factorising $x^3 - a^3$) the roles of x and a are the same, that is both are variables as described above. I could argue that this task is exactly equivalent to factorising $a^3 - b^3$. However the very fact of using x and a instead of a and b relocates the task into a different context, that is the context of polynomials in x , with its attendant connotations of functions, graphs and equations. For Tommy, in the context of having just worked on factorising $x^3 - 1$ and $x^3 - 8$, the roles of x and a are very different. By the end of my conversation with Tommy he still was not comfortable with the presence of a . I suggest that his comfort with x and discomfort with a are explained partly by the immediate context and partly by his familiarity with the use of x .

This example, then, highlights the stereotypical role played by x in many algebraic contexts. It is the generic unknown in equations to be solved and the generic variable in functional expressions, as well as being the independent

variable in the equation of a curve and first coordinate of the general point on a curve.

A general linear equation

A class of adult initial teacher education students was discussing the solution of linear equations. I asked them to give me 'a general form for a linear equation' and $ax + by = c$ was offered by Gill, a member of the group. Another member offered $ax + b = c$ and then $ax + b = 0$ to general approval. I asked

'These equations (*the ones we had looked at so far*) have had x s in them - they haven't had any other letter in them. Now the equation that Gill's brought up here ($ax + by = c$) has got a, b, c, x and y in it and some people are objecting to the y . Why are you objecting to the y and not the a, b and c ?'

After a few moments' pause there were two replies to this question:

'You're assuming that a, b and c are just ordinary numbers and x and y are the variables'

' a and b are used to stand for numbers that you know and x and y are numbers that you don't know'

Although it was not something to which they could recall having previously given any conscious thought, these students were in no doubt that a, b and c played different roles from x and y . In the ensuing discussion they described this as 'conditioning'. Some of them expressed surprise that they accepted this difference between roles without any good reason or conscious acknowledgement.

The role of x and y as the coordinates of a general point on a curve combines with the roles of x as the unknown in equations and as the argument of functions to set them apart from all other letters. These students' explanations of the differences are not entirely coherent but they are deeply felt.

Introducing k

Working on Problem N 'Find the equation of a straight line which passes through the point (m, c) ', a colleague wrote down the equation $y - c = k(x - m)$. He was concerned that he had written down a general formula when the equation of a straight line was asked for. He said

'So I thought "well I've introduced the variable k " - it's funny it didn't bother me that I'd introduced y and x - but I thought "I've introduced k " which I didn't like'

For me it was no surprise that he was not concerned that he had introduced y and x . In a sense he had not introduced them, rather they had introduced

themselves, because there was no moment of choice concerning which letters to use. The universal use of these letters as 'default' variable names in the equation of a straight line make it possible to use them without experiencing choice.

Similarly, Tommy was not unsettled by the presence of x . For him x was *the* variable. The meaning of a , by contrast, was still mysterious.

Each of the examples above, under the heading 'What is special about x and y ?', points to some aspect of the unique roles played by x and y in our mathematical culture (by this I mean, in particular, the culture represented by teachers and examiners of 'A' level mathematics in England and Wales, and into which pupils need to be, to some degree, inducted. Many of the features of this culture are common to other groups). The very strong cultural pressure to use x and y in the circumstances exemplified above makes it almost a mathematical necessity. Consider for instance, the task

'Find the equation of a straight line which passes through the point (x, y) '

Responses of the form $y = mx + c$ hold on to the conventional roles of x and y whilst making their new roles, suggested by the question, as unknown particulars, untenable. Responses of the form $Y = m(X - x) + c$ relinquish the expected roles of x and y in order to have them adopt others.

M or m? Particular or General?

During June and July 1994 I asked a number of students individually to work on

Problem D Find the equation of a straight line which has gradient M and passes through the point (p, q)

I had deliberately given M as the gradient rather than m so that the stage in the working of substituting the known gradient (M) for the general gradient (m) in the equation $y = mx + c$ should be apparent both to the students and to me.

Most students used the two letters as I expected, stating first either $y = mx + c$ or $y - y_1 = m(x - x_1)$, and following this by substitution of M for m . Two students, Frank and Paul, however, made no distinction, in what they wrote or said, between the gradient given in the question and that used in the 'template', using m throughout. There was, in any case, no distinction between lower and upper case ms in Paul's handwriting. Their not seeing the distinction between M and m suggests three different interpretations to me.

One is that m is so familiar as the gradient of a straight line that it is used as such automatically, without any decision having been recognised. It is possible

that they did not 'see' the reference to M in the question in the same way that Naomi did not 'see' the roles of m and c in the question I set her, and the students at school A did not 'see' p and q in the test question.

A second is that the distinction between general and particular, that is between the placeholder role of m in $y = mx + c$ and the particular-but-unknown role of M , is not evident to Frank and Paul and therefore they see no need to mark this distinction by use of different symbols. I will say more about this distinction below.

A third is that they see no distinction between lower and upper case letters in mathematics. This is a common feature of the work of some students, so that, for instance the equation for a general quadratic might be written $y = ax^2 + Bx + C$. I noticed this feature in Paul's work on several occasions but never in Frank's work other than on this problem.

Symbol choice is not arbitrary

Speaking of the role of pronouns as placeholders and, in comparison, that of literal symbols in mathematics, Wagner (1979) points out that from a mathematical point of view the choice of symbol makes no difference, whereas there is a difference, in language, between the use of, say, *he* and *she*. In other words the choice of pronoun as placeholder conveys some information about the noun which it has replaced, whereas in mathematics the choice of letter does not necessarily convey information about the quantity for which it stands. Whilst this may be true in the strict mathematical sense, it is by no means the case from the cultural point of view. I have already made a distinction between mathematically necessary and culturally determined choices. Attempts to establish conventions concerning the meaning conveyed by the choice of letter are not new. As early as the 1580s Vieta's use of consonants for given quantities distinguished them from unknowns which were represented by vowels.

The examples I have given show that choice of literal symbol can convey a great deal about the role of the quantity that it represents. In particular the letters x and y carry with them a great many contexts, meanings and metonymic triggers.

The teacher's dilemma

The extent to which students have stereotyped expectation of the role of certain letters is in part influenced by the teacher's choice of learning contexts.

This theme was taken up by the teachers' group with whom I met. One of the teachers, John, was very interested in a mathematical problem called 'Tangential' (Problem S) which is examined in some depth in chapter eight and

which suggested to him the theme of using alternatives to x and y . He returned to this theme on a number of occasions during the next meeting.

'If you use x and y , small x and small y as a sort of standard thing to use you actually get yourself into quite some difficulties and you've got to think, you've got to invent some different symbols, capital X and capital Y or something or h and k or whatever so in a sense you've got to use the symbolism, be aware of the symbols and what they mean.

'When I wrote it (*a solution to the Tangential problem*) down for about the third time and eventually tried to get it so that the symbols were consistent so that it actually worked ... it seemed to me it was actually quite an important thing for us to think about when we were doing the ordinary type of question if you see what I mean.

'There's a similarity to questions in the sort of older 'A' levels where you were trying to find the locus of a point. It would be in coordinate geometry and you would have some sort of chord to a curve or something and you have to find the locus of the mid-point of the chord and you'd have to specify "let (X, Y) or (h, k) " or whatever you want to do.

'Maybe when one is dealing with a situation with points on a curve maybe one shouldn't always just use x and y as a normal thing. That's the thought that I had as I was writing this out. Maybe one should use some other symbol sometimes just to sort of - you get the situation when you've actually got to substitute for the curve, you know $x^2 + y^2 = 2$ or something, you've actually got to substitute in for each, and you put in $h^2 + k^2 = 2$ or something, to make it obvious what you are doing.

'I was thinking a bit more in a sense of that you're not actually labelling - it's a general point - but on your diagram at that particular time it's a particular point that you make general and if you don't call it (x, y) initially but call it something different then maybe if you get used to doing that you're actually not going to have such a problem when in fact you're dealing with a point on the curve.

'Just thinking of the curve $y = x^2$ how often do we write it down for students that $b = a^2$?'

John had used his experience of working on the task which I suggested to prompt his thinking on a complex issue. He was concerned with points on curves which are in some sense 'general', but for reasons of clarity are best not labelled as (x, y) . For example in the Tangential question, there is a need to consider the point on the curve $y = e^x$ at which the tangent through the origin

touches. Using x and y for the coordinates of this point is potentially very confusing because of the use of x and y in the equation of the curve itself and the equation of the tangent. The kind of 'old 'A' level question' that John describes would be similar to Problem O, where the task was to find the locus of the mid-point of a chord. In the question he envisages the equation of the curve is not expressed parametrically so there is no obvious choice, other than x and y , of symbols to represent the coordinates of a general point on the curve. Again it would be potentially confusing to use x and y here because they are used in the equation of the original curve. However, I think his main point is that students should be alerted to the fact that the choice of x and y is in some sense arbitrary, so that they are empowered to make a different choice if it would be useful to do so. David, in his remarks quoted in part above, makes the opposite point:

'If you stick with x and y they know that - because I was thinking - they were differentiating and suddenly one of the examples, instead of being to do with θ or x , was e^{kt} . And I looked at it and I thought "how many of them are going to ask 'which do I differentiate it with - t or k ?' " But they've obviously come across enough examples, very few but enough, to go for the t and not the k - or they've come across k being a constant enough. So they didn't ask, which was a surprise to me, but it shows that if you do stick with always using the same things then at least they understand the more ordinary situations - they can make an educated guess - if you start using any old letter I'd say it's more likely to confuse most of the time.'

These two points of view illustrate the tension for the teacher between establishing the conventions of mathematical society and exposing them as culturally but not mathematically necessary.

On the one hand, my practice of cultural conventions in the use of letters allows me to automatise procedures. I can perform a procedure without placing my attention on that procedure. The role of each quantity in the procedure is captured by the name, that is the letter I use. I do not need to ask myself (for example) 'why was I trying to calculate c ?'. I know that the value I have calculated is the value of the y -intercept. My attention is not on the meaning of c and can therefore be on some other aspect of the problem. These conventions can also assist students in dealing with what Adda (1982, see chapter two) refers to as 'homonymy', that is the different roles of letters within the same equation. In her example, ' $ax^2 + bx + c = 0$ ', the roles of a , b and c are in fact separated from that of x by conventional usage so that distinguishing between them is not an apparent difficulty for students.

On the other hand, the repeated use of convention in symbol choice makes the cultural nature of the conventions invisible. It removes from view the choice of letter, so that the distinction between convention and mathematical necessity is blurred. The automatisisation of procedures is useful precisely because it removes attention from that procedure. The drive to automatise through rehearsal may remove attention too soon from where it is needed.

Conventional use of letters is a means of control for the expert user. These users can free their attention from the routine to place it on the unfamiliar. They also have the option of not using the conventional letters if they wish. The novice, by contrast, is controlled by the choice of letters. Their ability to perform a task may depend crucially on its being expressed in terms of the conventional literal symbols or on its being possible to perform the task by using the familiar notation. For example, in each of these pairs of tasks the second is much more difficult than the first for most students:

- A (i) Differentiate $\sin^{-1}x$ with respect to x
- (ii) Differentiate $\sin^{-1}h$ with respect to h
- B (i) Find the locus of a point which moves so that it is equidistant from the point $(1, 2)$ and the x -axis.
- (ii) The point (a, b) is equidistant from the point $(1, 2)$ and the x -axis. Find an equation linking a and b .

Task A(i) requires me to repeat a procedure which I have rehearsed. Task A(ii) requires me to be content that the procedure is still valid if the letter with which I have rehearsed it is replaced by h , and to be secure enough in my procedure to be able to use the replacement.

Task B(i) requires me to rehearse a procedure. Task B(ii) requires me either to work on this as an unfamiliar problem or to recognise it as a familiar problem in an unfamiliar guise and make a translation.

Students' difficulties in dealing with familiar tasks couched in unfamiliar letters is testimony to the extent to which their concepts are tied to notations. The challenge for the teacher is to harness the usefulness, in terms of fluency and automatisisation, of the use of conventional symbols, whilst at the same time offering the opportunity for students to work on their awareness of the cultural nature, as opposed to mathematical necessity, of these conventions.

The existence of stereotypical roles for certain letters is another aspect of the complexity of expressing relationships between variables and of understanding the shifting roles of those variables.

Unknown and Given

The previous section has established that certain roles are very commonly associated with particular letters in their use as literal symbols.

One way to distinguish between roles is to ask which quantities are to be determined and which are to be assumed known. In the situations which I describe in this section students were working on questions which required them to use literal symbols representing quantities which were to be assumed known. In the first episode the emphasis is on using known values of the variables x and y to find values of the parameters m and c . Because there is insufficient information to find both values, one of the two must be assumed known.

Variable and Parameter

Paul and Trevor had asked me to give them a revision session on aspects of coordinate geometry because both had missed some of the lessons on this topic. I began by speaking to them about the general form for the equation of a straight line, $y = mx + c$, and how they would use it to find the equations of particular lines. I asked them to explain what they could tell about the equation of a straight line if they knew the line went through the point $(1, 2)$, and, after their answer, I continued by asking them what they could say about the equation of a line which passes through $(0, 4)$ (the transcript extract is from 'Trevor and Paul I' lines 55 to 74).

Liz: I'm going to say now it goes through $(0, 4)$.

Trevor: As well as $(1, 2)$.

Liz: No.

long pause while Paul and Trevor write

Paul: It's $(4, 0)$. *(This reversal of the coordinates of $(0, 4)$ is treated as an oversight by me)*

Liz: Yes.

Trevor: c has to equal 4.

Paul: c equals 4, because 4 equals $0 + c$

Trevor: So the gradient is 0.

Liz: It tells us that c is 4, which is, you could have done that by a slightly different sort of reasoning because, c , you said to me was the point on the y -axis where it cuts.

Trevor: Yeah.

Liz: And this point $(0, 4)$ is on the y -axis. It goes through $(0, 4)$ then c is 4. What does it tell us about m ?

Paul: That it's 0 because

Trevor: I don't know if it would be 0, cause you are just saying that x is 0. It still could be at an angle

Paul: We know, we know that $y = 4$, in this particular case and we know that 4 is c , so we know that mx has got to equal 0.

Liz: Yeah.

Paul: And the only way mx could equal 0 is if m is 0.

Trevor: No but

Paul: No, No, No - because x is 0.

Trevor: x is zero, so m could be anything

Paul: Yeah that's it so m could be anything

Trevor and Paul's attention throughout this extract and most of the rest of the conversation was on substituting values for x and y into an equation for a straight line in order to draw conclusions about the values of the parameters, m and c . For example, immediately before the events portrayed in this extract took place they had substituted $x = 1$ and $y = 2$ into $y = mx + c$ to obtain $2 = m + c$ and said

Trevor: $2 = m + c$ $m + c$ there's two unknowns

Paul: Well you know $m + c = 2$

They began by treating the point $(0, 4)$ in the same way. My agenda was different. I wanted them to see that $(0, 4)$ can be treated differently because it is on the y -axis, but my intervention failed to shift their attention away from the substitution they had made. In response to my question about the value of m they returned to their equation $y = 0 + 4$ and Paul deduced that m must be equal to zero. He said '.. mx has got to equal 0', 'And the only way mx could equal 0 is if m is 0.' One interpretation of his line of argument is that he has lost sight of the substitution that has been made for x . This substitution effectively turned x into a given and m into an unknown, so that the students were seeking a value for m . Paul, knowing that he was seeking information about m , chose to treat x as indeterminate, a varying quantity which must be given freedom to vary, rather than treating it as a known value, 0.

The situation was compounded by the fact that there was insufficient information to calculate a value for m . The students may have expected to be

able to find the value of m (otherwise why would I have asked them 'What does it tell us about m ?'). Taking x as indeterminate rather than given enabled them to do so.

This incident is also an example of what Furinghetti and Paolo (1994) speak of as students' difficulties with letters in apparently symmetrical roles i.e. $mx = 0$. The confusion might be understood as one between the two different roles of the letters in an equation which gives no clues as to which letter is playing the role of unknown and which the role of given.

Unknown-to-be-taken-as-given

The distinction between unknown and given became salient to me when I looked at the work of a number of students on the following problem:

Problem J Find, in terms of a and b , the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$.

In this problem the 'given' values are not numerical but stated as literal symbols, a and b . These are to be taken as given, even though they are not 'known' in the sense that their numerical value is not determined. The variables in this question, x and y , need to be 'found' as expressions involving other letters.

The role of 'unknown-to-be-taken-as-given' is a further example of a role somewhere between particular and general.

In the four transcripts which follow I find very different kinds of understanding concerning the roles of these 'unknowns-to-be-taken-as-given'. Through the students' dealings with this question I explore what is involved in an expert awareness of the role.

Eddie

I have discussed an early part of my conversation with Eddie on this task in chapter five. We had established that he needed to find the point where the given line and the perpendicular met. In what follows he combines the equations of the two lines by eliminating x to give him expressions for a in terms of b and y , and for b in terms of a and y (the transcript extract is from 'Eddie' lines 89 to 113).

Eddie Well I'll do some working out

Liz Okay

Eddie Well in this one we put it in terms of x so $x = 4 - 2y$, and on the second one you just use your equation $y - y_1 = m(x - x_1)$ - using a

and b as the points which you're given so you've got $y - b = 2$, which is the gradient we've already found out, $x - a$ - which is $y = 2x - 2a + b$ and as I've got x as the subject of the formula, divide through by 2 which gives us $\frac{y}{2} = x - a + \frac{b}{2}$. Take x to the other side of the bracket then times through by negative 1 which gives me $x = \frac{y}{2} + a - \frac{b}{2}$. So from that I'd use, I've found out $x = 2y$ on the first one so I put $4 - 2y$ is equal to the other side of the x on this, which is $\frac{y}{2} + a - \frac{b}{2}$ times both sides by two to get rid of the fractions, which will give me $8 - 4y = y + 2a - b$ and we'd like to find it in terms of a and b . So first I make b the subject of the formula so $b = 5y + 2a - 8$ and then make a the subject of the formula, first of all taking over the $2a$ so $2a = b - 5y + 8$ and then divide through by 2 which will give me $a = \frac{1}{2}(b - 5y + 8)$ to eliminate the fractions

Liz Uhmhm. Okay.

Eddie It looks a bit messy (*laughs*)

Liz What is it you've worked out?

Eddie Er, the coordinates of a and b

Liz Right, that's um, not strictly what they asked you for.

Eddie No so I'd have to find the coordinates a intersects, putting x and y as the origins, I think ...

Liz Putting x and y as ..?

Eddie As the parameters, like I'd have, make x the subject of the formula as in this case. I've got this equation, say, take this one for example $8 - 4y = y + 2a - b$ just make y the subject of the formula, then find out the y -coordinate, then sub it back into the first one which seems the easiest $x = 4 - 2y$ to find the value of x , so just go like this $5y = 8 - b - 2a$

Liz Yeah, I think it's *plus b*. I don't know which one you were doing it from

Eddie $5y$ take this over that side, put this over that side you've got $-5y = -b + 2a - 8$, times through by -5 , you've got $5y = b - 2a + 8$ yeah ... so you could divide through by 5 which will give

$y = \frac{1}{5}(8 + b - 2a)$. Then you could sub this in for y in the first one, I don't know if you could write that down, .. so then $x = 4 - 2y$ so it would be $4 - 2(\frac{1}{5}(8 + b - 2a))$, which is just $x = 4 - \frac{2}{5}(8 + b - 2a)$. I'll just work this out to make this easier, so that would leave $4 - \frac{16}{5} + (\frac{2}{5}(b - 2a))$ so you just do .., that's three and one fifth and four minus three and one fifth is four fifths, so $x = \frac{4}{5} + \frac{2b}{5} - \frac{4a}{5}$ so you can take the $\frac{1}{5}$ out again so $x = \frac{1}{5}(4 + 2b - 4a)$.

Liz Okay, so what would you give as the answer to the question?

Eddie I'd give, er, 'in terms of a and b ...' Well I've found the coordinates, where these cross in terms of x and y but you need to find it in terms of a and b .

At several points in the conversation Eddie's use of specialist mathematical terms was cavalier. This impression is given by his tone of voice as well as his words. For example he spoke of 'putting x and y as the *origins*' and when this was queried he replied with 'As the *parameters*'. Finally he re-used the language he had used a few lines earlier 'make x the *subject of the formula*'.

Amongst the general confusion over the meaning of terms it emerged that Eddie had understood 'in terms of a and b ' to mean that he should find expressions for a and b . His first attempt to answer the question, by expressing a in terms of b and y , and b in terms of a and y , seems to have been driven by a belief that the question expected expressions for a and b . He does not seem to have considered which letters should be allowable within these expressions. The appearance of b and y in the expression for a seems to have been the result of accident rather than choice. The meaning of an expression for a in terms of b and y was not questioned. There was no sense of expressing an unknown (the coordinates of the point of intersection) in terms of givens (a and b).

Lorne

Lorne needed a little help in solving the simultaneous equations which resulted in expressions for x and y in terms of a and b and I carried out some of the algebraic manipulation necessary. In fact there was a mistake in my working which went unnoticed at first. When we had achieved these expressions I asked him about what he had done (the transcript extract is from 'Lorne' lines 299 to 326).

Liz: Okay. This question talks about point (a, b) . And it asks you to give an answer in terms of a and b . Umm, what's, what's going on then? Tell me about this point (a, b) and this point that we've found.

Lorne: Well once you've got a point (a, b) like if that's $(1, 2)$, then afterwards you can work out what that is because you can, you've just got a and b in there, so you can work out the coordinates of where it meets then.

Liz: Right. Umm so could you work out the coordinates of the foot of the perpendicular from the origin to that line?

Lorne: From the origin to that line?

Liz: Yes.

Lorne: Umm, umm, it's, ... well yes you can because you get your coordinate (a, b) , even if it's up here, then afterwards it's a along and b , so it's just use Pythagoras there, and you work out, you're saying the length of that from the origin, the origin's here. Yes?

Liz: No. Listen again.

Lorne: Okay I'll listen.

Liz: Can you find the foot of the perpendicular

Lorne: Uhmhm

Liz: from the origin to that line?

Lorne: from the origin to that line?

Liz: Uhmhm.

Lorne: Does it work out a triangle?

Liz: Can you draw in the perpendicular from the origin to that line?

Lorne: The perpendicular from the origin, to this line?

Liz: Yes.

Lorne: No, *(draws in the appropriate line)* is that what you're saying?

Liz: Yes.

Lorne: Uhmhm.

Liz: Now if we wanted to work out the coordinates of that point, can we use the working that we've done in this question to help?

Lorne: Yes you can. You can because it's going to be $-\frac{4}{5}$ - oh it can't be $-\frac{4}{5}$

The expression which we had worked out for the x -coordinate of the foot of the perpendicular gives a value of $-\frac{4}{5}$, but Lorne's diagram made it clear that this was not correct. The discrepancy between what appeared on his diagram and what was predicted by the expressions we had obtained then alerted us to the mistake in my working. We went on to correct the mistake.

Lorne stumbled over making use of his rules, perhaps because his visual image of the geometrical situation was not very clear, or perhaps because the point I chose (the origin) was not part of his set of imagined possibilities for the point (a, b) . His initial answer ('Umm, umm, it's, ... well yes you can because you get your coordinate (a, b) , even if it's up here, then afterwards it's a along and b , so it's just use Pythagoras there, and you work out, you're saying the length of that from the origin, the origin's here. Yes?') suggests that he was trying to find the distance from the origin to (a, b) rather than seeing $(0, 0)$ as a particular instantiation of (a, b) .

However his idea of the role of (a, b) in this situation was quite clear. He knew that what he had found were rules for working out the coordinates of the point of intersection from the values of a and b .

Robert

I asked Robert to comment on the form of the answer to the question (the transcript exact is from 'Robert' lines 82 to 84)

Robert: Why is it going to contain a and b ?

Liz: Yes.

Robert: ... Well, could be cos it depends on where the point is, I mean obviously the point is a long way away. You need to know where the point is. You'll never find the distance from the point to a line if you don't know where the point is.

To Robert it was obvious that the answer to the question must be 'in terms of a and b ' (his tone was quite sarcastic in the sentence 'Well could be because it depends on where the point is'). He was quite clear that this meant that the answer contained a and b , and, as he showed later in the conversation, in what sense the answer depended on a and b (although he mistook what it was the question was asking for, speaking as though the perpendicular distance was required rather than the foot of the perpendicular).

Trevor

Trevor worked on the same question. He reached the point where he had found the gradient of the perpendicular line and wanted to find the constant

term by substituting a pair of coordinates of a point which the line passed through. He was using the form $ax + by + k = 0$ rather than $y = mx + c$. (the transcript extract is from 'Trevor and Paul II' lines 174 to 180).

Trevor: You don't have a specific point 'cos if you could put in a point

Liz: Well you do have a specific point .. 'cos you know the line goes through (a, b)

Trevor: (a, b) .. yeah so I could put (a, b) in - so it would be - yeah but it would be $2x + y$ and you don't know what k could be - it could be a negative or I don't know .. it's weird

Liz: So if you want to find out what the value of k is

Trevor: You need a coordinate so you can put the numbers in so you can prove that that is zero and then you can find that value of k but if you put a and b in you are just going to come out with

Liz: something in terms of a and b

Trevor: Yeah

Trevor knew that there was a procedure he could use to find the constant term in the equation of a straight line if he knew a point that the line went through. This procedure was well rehearsed and familiar. But he remained to be convinced that knowing that the line went through (a, b) amounted to knowing a point. He admitted that he could substitute a and b for x and y but thought that this would not tell him what k was. He still came back to 'You need a coordinate ..'.

On other similar occasions during the same conversation Trevor made a number of utterances which give clues about his thinking. Having used the condition that the line $y = mx + c$ should pass through $(1, 2)$ he said

' $2 = m + c$ $m + c$ there's two unknowns' (Trevor and Paul I line 29)

After arriving at the equation $c = b - 2a$ he said

'There is too many unknowns so you have to leave it like that (*laughs*)'
(Trevor and Paul II line 144)

Wondering how to proceed with the question he then said

'So you can... I don't know ... you could work it out without using c I suppose'
(Trevor and Paul II line 148)

Asked why he wanted to find c he replied

'I don't know .. if you find out the point of c it might give you the place where it crosses the y axis but there's no point 'cos you need b and a anyway wouldn't you.' (Trevor and Paul II line 153)

Frank also was faced with the situation of having one variable in terms of others which were to be taken-as-given.

'It's because c is if I was working it out with numbers c would be $q - mp$ But as I don't know what $q - mp$ is, that's the closest I'm going to get to what c is' (Frank lines 482 to 486)

' q take away mp , (writes $c = q - (mp)$)..... this one - it's knowing q to take away from it' (Frank line 466)

For Trevor and Frank, expressing one letter in terms of another or others was of little benefit.

Trevor : there's no point 'cos you need b and a anyway wouldn't you

Frank: it's knowing q to take away from it

Such an expression would not convey any useful information. They do not see any use for expressing the way in which one variable depends on another. To constitute an 'answer' the expression must be numerical.

I might conceive of Trevor's and Frank's understanding as being at a particular stage of development by comparing it with the models suggested by Kieran (1994) and Sfard and Linchevski (1994). Their struggle with problems of the kind they were working on above suggests that they are not yet working at the level of functional algebra, where relationships between variables are the focus of attention. Rather they want to work at the level of finding the value of an unknown, that is at the level of algebra of a fixed value.

However such an analysis does not sum up for me the nature of their experience in facing these questions. The questions require that an answer be given in terms of a and b , or p and q , that is the answer to the question will be an expression of a relationship between 'to-be-taken-as-given' variables, and 'unknowns'. In Trevor's question the unknowns are labelled x and y and are the coordinates of a point of intersection. In Frank's case the unknowns are expressed by the placeholders m and c in the template $y = mx + c$. But neither of them sees the gain in finding an expression that will link k with a and b , or c with p and q . Such an expression conveys no information to them. They do not 'know' k by expressing it in terms of a and b , or c by expressing it in terms of p and q .

This is in contrast with the experienced mathematician who hardly distinguishes between 'knowing' one variable in terms of others and knowing its numerical value. For example, when working on a conical pendulum problem I might use Newton's laws of motion and my knowledge of acceleration in circular motion to derive the equation $T = ml\omega^2$. Having done so I would say that I 'know' T . By this I mean that I know how it is connected to other variables in the situation, that I could calculate a numerical value if I knew the numerical values of m , l and ω , that I have some sense of control over T . (Of course, I may generate several equations linking these and other variables and be unsure as to how many independent connections have been established. Nevertheless, connections between variables appear as potentially adding to my knowledge about the situation).

Another pair of phrases in Trevor's and Frank's utterances were particularly striking:

Trevor: you have to leave it like that

Frank: that's the closest I'm going to get to what c is

Both betray that the 'answer' that has been achieved is unsatisfactory in some sense, but the best answer available. Both are reminiscent of the language patterns used in describing situations which arise earlier in the mathematics curriculum, in basic algebraic manipulation. For example, on arriving at an answer of $3 + 4a$, a teacher or pupil might comment 'you have to leave it like that' (and not, for instance, change it to $7a$). Robinson et al (1994) quote some Israeli pupils (in translation) who were working on these types of problems

'It should remain as it is' p131

'The answer stays $4x + 6$ ' p132

The desire for 'closure' is well documented (Collis 1974, Booth 1988, Kieran 1992). The language patterns above are those of a learner who is overcoming their intuitive desire to 'finish' the question by, say, adding together $4x$ and 6 , or of a teacher who has recognised this intuitive desire in their pupils. They are not the language patterns of an expert user who would not feel any compulsion to add the two terms. These utterances suggest a learner who has begun the process of coping with 'lack of closure' but who is still repeating a received wisdom rather than feeling it to be their own.

Similarly Trevor and Frank have acquired the phrases 'too many unknowns', 'you have to leave it like that' and 'that's the closest you're going to get'. They have not reached the stage where achieving an expression in terms of other variables carries a feeling of 'having found'.

The difference between the levels of control over this problem which I perceive to have been experienced by these various students is strongly connected with awareness of relationship between quantities. Trevor did not conceive of the expressing of such relationships as a legitimate aim. Eddie was unclear about which quantities are to be expressed in terms of which others and what was the meaning of such expressions. Lorne was aware that he was providing a general rule for calculating the coordinates of the point of intersection given the values of a and b , but he did not immediately use his rule when presented with a situation in which he might do so. Robert was amazed by the triviality of my question. For him it was perfectly plain that the position of the foot of the perpendicular must depend on a and b . His answers showed that he has mentally connected the algebraic and geometric aspects of this dependence.

These four students, then, demonstrate very different levels of awareness of the nature of functional dependence. Such awareness is fundamental to successful dealing with this and many similar situations in 'A' level mathematics.

Teaching approaches need to take account of the need for this awareness as well as the need to develop skills in, say, finding the gradients of perpendicular lines and solving simultaneous equations.

Shifts of Role

Within this chapter my theme has been the complexity and ambiguity associated with roles of variables. This ambiguity has sometimes manifested itself as a shift in the role of a variable during a problem solution. In the problems which I have discussed, this shift took one of four forms. Of these four the first shift involves the letters x and y , whereas the other three involve different letters. I have already established that these letters are treated very differently from others in 'A' level mathematics.

The first shift is in the role of x or y from variable to unknown-to-be-found. For example, in a solution to the question 'Find the coordinates of the point where the line $x + 2y - 4 = 0$ meets the line $y = 2x - 2a + b$ ' x and y are first seen as variables whose importance is in their relationship to each other. Each can take any real value. However as soon as the learner begins to solve these as a pair of simultaneous equations, x and y take on the roles of unknowns, whose numerical values are to be found. Other examples of problems involving this shift are questions which require the coordinates points of intersection with the axes or of turning points. Although it is a significant issue in earlier years at school, this shift in role seems to cause very few problems for 'A' level students and is almost universally glossed over.

The second shift I have identified is from the role of placeholder within a form to that of unknown-to-be-found. For example in answering the question 'What is the equation of a straight line with gradient 3 which passes through the point (2, 8)?' a student might substitute $x = 2$ and $y = 8$ into the form $y = 3x + c$ to find a value for c . As a result of this substitution, c changes from being a placeholder for 'the y intercept' within a standard form, to being an unknown-to-be-found.

The third shift is from unknown-to-be-taken-as-given to unknown-to-be-found. This shift can occur when an analytic solution method is used. For example, in solving analytically the problem 'Find the point of contact of the tangent to the curve $y = x^2 + 1$ which passes through the origin', the first step is to name the unknown by choosing a letter to stand for the x coordinate of the point of contact. The chosen letter, a say, is then treated as given and used to form equations which express relationships between a and other quantities. Finally those equations are solved by treating a as unknown-to-be-found.

The fourth shift takes place when a quantity which was originally conceived of as constant, though unspecified, is allowed to vary, that is it is a shift from unknown-to-be-taken-as-given to variable. This shift frequently occurs in solutions to locus problems, for example 'A point P, coordinates (a, b) is equidistant from the x -axis and the point (3, 2). Find a relationship connecting a and b .' In the solution to this problem, a and b are first taken to be fixed but unspecified, so that expressions for the distances from (a, b) to the x -axis and (3, 2) can be formulated in terms of these unknown-to-be-taken-as-givens. Once these expressions have been equated the equation formed can be seen as a relationship between variables and a and b can be allowed to vary in order to map out a parabola. In this question this last stage, which represents the locus aspect of the problem, is not emphasised, because the question asks merely for a relationship between a and b . An emphasis on the locus aspect of the problem is usually accompanied by a change in notation which allows the final relationship to be expressed in terms of x and y . This notational change allows the shift to seeing the letters as variables to take place more easily because the conventional roles of x and y are as variables.

Each of these four shifts represents an awareness which has become unconscious for the teacher but which may not yet be present for the student. Without such awareness, problems such as those in my examples can only be solved by mechanistic application of rules.

Roles of Literal Symbols

This chapter has pointed to three areas of awareness concerning the roles of literal symbols. The first is that the different roles played can be conceived of as

different places in the order of variation. Different images of the problem situation will result from different hierarchies. The second is that certain letters are very strongly associated with certain roles in the minds of students and expert users. This association has both useful and counter-productive outcomes. The third is that the role of unknown-to-be-taken-as-given is particularly problematic for some students. Their experience of expressing one variable in terms of others is not adequately described as 'finding' that variable.

In each of the three sections I have looked at the problems which arise for students because of the complexity of the roles of variables which lie somewhere between particular and general. In each case the ability to deal with this complexity is characterised by a flexibility of approach. Use of different **orders of variation** can give access to different aspects of a problem situation and flexibility is required to move between one approach and another. To deal successfully with **stereotyped roles** the student must be fluent in the common usage of the literal symbols but flexible enough to use them differently where appropriate. In order to handle **unknown-to-be-taken-as-givens** appropriately students have to treat them in some respects as though they were variables and in others as though they were numerical values.

In this chapter I have focused on the roles played by literal symbols within problems. In the next I will look at the whole process of problem solving in the context of second variable problems.

Chapter 7 Patterns of Problem-Solving

Much of what I have observed in the mathematical work of myself, students and colleagues has been in the context of problem-solving. As I have witnessed individuals' problem-solving processes I have noticed several patterns in their actions which I connect particularly with the complexity of the roles of variables within the problem situation. I describe three of these patterns (losing track, 'solve and substitute back' sequence, and using forms) in this chapter. The chapter closes with a more general look at the notion of 'form'.

Losing Track

Three Accounts

To begin this section I present three accounts of working on mathematical tasks. The first is a transcript of a conversation between myself and two students. I have referred to part of it already in chapter six. The second is a transcript of my conversation with another student. This is followed by an account of my own work on a mathematical problem which presented itself to me in the course of a conversation with a colleague.

I suggest that you pause for a few moments to work on the mathematical problem yourself in each case before proceeding to read the rest of my account.

Finding the equation of a straight line

I was working with two students who had come to me for help with the chapter on coordinate geometry in their first year sixth form text book. We had been working about half an hour already and had come finally to a question in the last exercise in the chapter. It read as follows:

Problem J Find, in terms of a and b , the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$.

Trevor worked out that the gradient of the line $x + 2y - 4 = 0$ is $-\frac{1}{2}$ and drew a sketch of it, and we established that a first step would be to find the equation of a line perpendicular to this one which passed through (a, b) . (The transcript extract is from 'Trevor and Paul II', lines 137 to 153)

Trevor: So I want to change that around so the gradient is 2 for that line.

(Writes $y = 2x + c$) I know that the coordinates are (a, b) , so you have got ...

Paul: b .

Trevor: Yeah. b equals $2a$ plus c . Yeah?

Liz: Yeah?

Paul: Yeah.

Trevor: There is too many unknowns so you have to leave it like that.

Liz: That's a good point, there are too many unknowns in this question, for you to be able to get an answer with no unknowns in it.

Paul: That's why they want it in terms of a and b .

Liz: Yeah, that's right.

Trevor: So you ... I don't know ... you could work it out without using c I suppose

Paul: Well, c equals $b - 2a$ doesn't it?

Liz: Uhuh

Trevor: Yeah ... You could use that and that I suppose and put that there, but .. (He indicates putting $c = b - 2a$ into $b = 2a + c$)

Liz: Why were you trying to find out c ?

Trevor: I don't know .. if you find out the point of c it might give you the place where it crosses the y axis but there's no point 'cos you need b and a anyway wouldn't you.

What happens to the turning point?

I asked Sam to work on

Problem M Sketch $y = x(x - a)$

and he did so fairly easily. The transcript shows how the conversation continued (the transcript extract is from 'Sam' lines 94 to 116).

- Liz: Right, okay. What's umm - what effect does it have in general if you alter the value of a ?
- Sam: .. It's just going to widen out the parabola. The parabola's always going to go through the origin no matter what a is so it would either be more squashed together or more spread out.
- Liz: Right. Could, would you be able to tell me about what happens to the turning point if we alter the value of a ?
- Sam: Umm as a gets larger the turning point, the y coordinate gets smaller somehow, .. or would it stay the same? Could I try it with two points, with a point, a value for a Just need to differentiate (he writes $a = 2$, $y = x(x - 2)$, $y = x^2 - 2x$, $\frac{dy}{dx} = 2x - 2$, $2x - 2 = 0$, $x = 1$.
 $a = 4$, $y = x(x - 4)$, $\frac{dy}{dx} = 2x - 4$, $x = 2$) Umm what is this telling me? that's not telling me anything (he crosses out all of his working)
- Liz: What is it telling you?
- Sam: Well it's just giving me the x point, x coordinate.
- Liz: Okay and what, what do you want.
- Sam: Ah, y oh yes, if I sorry, if I put the x coordinate in and then 2 times then it's 0.
- Liz: What are you putting in?
- Sam: .. x equals 4.
- Liz: .. Where's that coming from?
- Sam: No, x equals 2, sorry, that'd be (he writes $y = 2(2 - 4) = -4$,
 $y = 1(1 - 4)$, $y = -3$)
- Liz: Umm this is the case where $a = 2$ isn't it?
- Sam: Ah hmm.
- Liz: You've got $x = 1$ there.
- Sam: Oh yes. $1 - 2$ (he replaces the 4 in $y = 1(1 - 4)$ by a 2 and changes the final answer to -1) that's -1 , yes that's what I thought. As umm as a gets further away from the origin umm the turning point is going to get closer to the y , x -axis.

Liz: ... Is it?

Sam: no. Further away sorry.

Liz: Hmm. It actually gets deeper doesn't it?

Sam: Yes.

Affine transformations task

(I refer to this task as Problem G in chapter five, where I describe how I used a particular case with a group of teachers)

An affine transformation is a transformation of the form

$T:\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$ An important feature of an affine transformation is that it maps straight lines to straight lines. How can I prove that?

I want to show that all points on a particular straight line will be mapped to points on a (different) particular straight line. I will express the coordinates of the first point as $(x, mx+g)$. This ensures that this point lies on a particular straight line (with equation $y = mx + g$). So the image of this point is

$$\begin{aligned} T\begin{pmatrix} x \\ mx + g \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ mx + g \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \\ &= \begin{pmatrix} ax + bmx + bg \\ cx + dmx + dg \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \\ &= \begin{pmatrix} ax + bmx + bg + e \\ cx + dmx + dg + f \end{pmatrix} \end{aligned}$$

Now what should I do with this string of letters? What is the status of each? Which ones should I be concentrating on and which are 'noise'? What would it mean to show that this new point lies on a particular straight line? My new point should be 'of the form' $(x, mx + c)$, so should I be worried that the x -coordinate is not x ? Or can I take $(a + bm)x + bg + e$ as my new x and show that $(c + dm)x + dg + f$ is a linear function of it? Linear in what? This would involve constructing a coefficient of $(a + bm)x + bg + e$ and calculating a 'remainder'. I'll do that.

$$(c + dm)x + dg + f = \frac{c + dm}{a + bm}[(a + bm)x + bg + e] + (dg + f) - \frac{c + dm}{a + bm}(bg + e)$$

This seems to be something I could have done with almost any expression. I have no sense of this process 'working' because of some feature of the transformation. Can I imagine a different sort of transformation for which this would not work? I find that most, if not all of the transformations that I

have had to work on as student or teacher have been affine, without my knowing it! Rotations, reflections, enlargements, shears and translations all fit the form given. So I must imagine something I don't have a name for, for instance

$$T: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

What happens if I apply this transformation to the point $(x, mx + g)$?

$$T \begin{pmatrix} x \\ mx + g \end{pmatrix} = \begin{pmatrix} x^2 \\ (mx + g)^2 \end{pmatrix} = \begin{pmatrix} x^2 \\ m^2x^2 + 2mgx + g^2 \end{pmatrix}$$

But $m^2x^2 + 2mgx + g^2$ is not a linear function of x^2 , so this transformation does not map straight lines to straight lines.

Looking again at the form of an affine transformation I can see that it is one in which the coordinates of the image are linear functions of the coordinates of the object. And now it is obvious to me that it will map straight lines to straight lines.

Drawing Parallels

My transcribing of the first incident was prompted by a recognition of a type of event which had become familiar to me during my years as a school teacher. I mentally labelled it 'losing track' because Trevor had begun apparently being confident in what he was doing but ended at best unsure how to proceed. I recognised the same phenomenon in Sam's work on the turning point of the curve. My later analysis of the incident involving Trevor allowed me to apply another label to it. I will describe and explain this below.

I recorded my own thinking in working on the affine transformation task because in the midst of it I was reminded of the experiences I thought I had witnessed in Trevor a few weeks earlier.

My aim in presenting them to you is to trigger recollection of your own teaching and learning experiences so that you can lay my experiences alongside your own.

The moment which stands out for me from my account of my own work on affine transformations is the moment of seeing the expression

$\begin{pmatrix} ax + bmx + bg + e \\ cx + dmx + dg + f \end{pmatrix}$ For an instant it was just a sea of symbols. I felt that I had lost my hold on the meaning of each of these symbols. There seemed to be too many similar letters together and the complexity was overwhelming. The arbitrariness of the choice of many of the letters added to the confusion. I had

not anticipated the form of the answer, or what kind of form I would need to show it equivalent to.

I recognise in Trevor's actions my own experience of having performed an algorithm (in my case transforming a pair of coordinates, in his case substituting a pair of coordinates into an equation) in the hope that the next appropriate action would become clear when this one was completed. Similarly Sam began by working confidently on a task he had set himself ('Could I try it with a value for a ?') and by performing some well rehearsed algorithms, that is expanding brackets, differentiating and solving an equation. However, when he got to the end of the algorithm the next stage was lost to him. His question 'what is this telling me?' and reply '..... that's not telling me anything' are also good summaries of my response to seeing the 'sea of symbols'.

I recognise in Trevor's expression, 'There is too many unknowns', the feeling that I cannot handle the complexity of the resulting expression. From having clear cut meanings distinct from one another the letters a , b , c etc. had become merged into an indistinguishable crowd whose members had all the same status.

In chapter six I argued that Trevor's utterance 'too many unknowns' is an acquired language pattern. Here I see it also as an expression almost of bewilderment.

Again in chapter six I wrote about Trevor's uncertainty of the roles of the letters in this problem, particularly the status of a and b which are to-be-taken-as-given. Here in parallel to my own experience I can see his expression as one of not knowing what to do next, not recognising any way forward in the results of my first moves towards solution.

These three accounts together illustrate the labels which I created for myself to say what it was that was evocative and memorable about them. My first label 'losing track' serves as a reminder of the state of not knowing the next step even though I began with a feeling that I knew what to do. My second label, 'of the form', derived from this group of events is discussed later in this chapter in the section 'The Notion of Form'.

My recognition of 'losing track' was simultaneous with the event in each of the cases I have described. My reaction to finding that Sam had lost track of what he was doing was to attempt to direct his attention to what it was he had found, and to what it was he needed ('What is it telling you?' and 'Okay and what, what do you want'). On this occasion my prompt was enough to allow him to continue.

My reaction to feeling that Trevor was unsure about his next step was to try to draw him back to his purpose in embarking on the piece of manipulation he had just performed ('Why were you trying to find out c ?) In this case my intervention was not enough to recall for him the procedure he had, in my view, set out upon. In fact there is no evidence that he was, at this time, seeing his actions as part of a well-defined procedure. His reply ('I don't know .. if you find out the point of c it might give you the place where it crosses the y axis but there's no point 'cos you need b and a anyway wouldn't you') suggests that his attention is on the difficulty presented him by the 'too many unknowns' and not on the next stage in the procedure.

My reaction could be described as trying to draw attention to the structure of the solution. My awareness of 'losing track' as a phenomenon allows me to contemplate other actions in future.

For example I might invite students to recognise the condition of 'losing track' in themselves. Such a recognition releases the energy which was trapped by the feeling of confusion. I might suggest they review their progress on the problem so far in order to get an overview of their work. Looking over their work after the event allows students to think about the structure or procedure without getting lost in the detailed manipulation. I might also choose to talk through the next stage as I see it, though this is only one possible course of action.

Keeping a Grip

As a *caveat* I include a snippet of conversation that occurred in the lesson I referred to in chapter one. The class was working on

Problem A For which values of k is $k(k - 1)x^2 + 2(k + 3)x + 2$ positive for all real values of x ?

As I walked around the room I came across a boy who was attempting to solve an inequality by a very convoluted method.

Liz Now, are you keeping a grip on why you are doing this?

Student No! (laughs) If I do that I'll get confused and I won't be able to do two at the same time okay I'll get my grip later once I've worked out .. this is my gripping point

This student clearly felt that he needed all his attention to deal with the immediate problem of solving the inequality, and that it was not possible for him to think about his strategy as a whole while he was solving it. I am sure that, when I work on problems, it is not possible for me to avoid situations

where I have 'lost track' of what I am doing or why. What I can do, though, is to recognise when it occurs and have a strategy for recovering my route.

'Solve and substitute back' sequence

Many of the problems which my students worked on involved finding a value or expression for a variable and then using it by substituting it into an equation. Often this equation was one which had been used already in obtaining the value or expression. For example, in solving the problem 'Find the equation of a straight line with gradient 3 which passes through the point (2, 8)' they used the 'template' $y = mx + c$. Putting 3, 2 and 8 for m , x and y respectively, gives a value of 2 for c . This value is then substituted back into $y = mx + c$, which is the equation which was used to derive it. A similar sequence may be enacted in solving two simultaneous equations in two unknowns. The value obtained for the first unknown is substituted into one of the original equations in order to find a value for the second unknown. Again the equation into which the value is substituted is one which was used to obtain it. I have labelled this sequence 'Solve and substitute back'.

The establishment of this sequence as routine can be an assistance in achieving fluency in certain problem solving situations. Its routineness implies that it is done automatically, that is with a minimal amount of attention to this aspect of the task. There are many examples from my year's work in the two schools of this sequence being used effectively by students.

However the automaticity of the sequence does not become obvious when it is used effectively. It is much easier to detect that a sequence has become automatic when it is used inappropriately. In some of Trevor's utterances in our transcribed conversation I find an indication that this sequence can become automatic in an unhelpful way. Having just used the template $y = 2x + c$ and the substitutions of a and b for x and y , he obtained $b = 2a + c$. He re-arranged this to obtain $c = b - 2a$ and then said

Trevor: Yeah ... You could use that and that I suppose and put that there, but .. *He indicates putting $c = b - 2a$ into $b = 2a + c$*

A little later he used the template $2x - y + k = 0$ and the same substitutions to obtain an expression for k .

Trevor: So k is $-2a + b$ which is the same as that there .. so you can now put in the value for k .. $2a - b + b - 2a$..so you'd have $2a$.. so yeah, that would prove that that equalled zero

In each case the sequence 'Solve and substitute back' dictated that the expression obtained should be 'substituted back', but Trevor did not choose the

most helpful equation to substitute into. The automation of the 'solve and substitute back' sequence, that is the reduction of the attention paid to this aspect, prevented him from considering where the substitution should be made.

At the end of the year Frank was working on

Problem D Find the equation of a straight line which has gradient M and passes through the point (p, q)

I referred to an earlier part of this conversation in chapter five. As I have mentioned in chapter six, Frank did not use M in his working, as suggested by the question, staying always with m . Having worked on the question for a time we arrived at the expression $c = q - mp$, which Frank then substituted into $q = mp + c$ rather than, as I had hoped, into $y = mx + c$. (this transcript extract is from 'Frank' lines 479 to 500).

Liz: Right. Yes. Tell me about that equation that you've just written down at the end there (*the equation reads $q = (mp) + (q - (mp))$*)

Frank: $q = mp +$ erh open brackets $q - mp$.

Liz: Yes.

Frank: Closed brackets. It's because c is - if I was working it out with numbers c would be $q - mp$. But as I don't know what $q - mp$ is, that's the closest I'm going to get to what c is

Liz: Okay. So you've worked out something for c . And we have to be satisfied with that. But what are we going to do with this 'something for c '?

Frank: Erh I don't know.

Liz: Have a look back at the one you did with numbers and see what you did with it.

Frank: Once I'd worked it out, I put it back in.

Liz: Hmm. Put it back in what?

Frank: Into the equation there. (*points to $y = mx + c$*)

Liz: Yes. (*during this long silence I wait for Frank to notice that he has not substituted into $y = mx + c$ this time. When he says nothing I take a different approach*) This is the equation of a line isn't it? (*pointing to $y = 4x - 5$*)

Frank: Yes.

Liz: Is this the equation of a line? (*pointing to $q = (mp) + (q - (mp))$*)

Frank: Uhmhm. Erh if it had the x and y in

Initially Frank substituted $c = q - mp$ into $q = mp + c$. This substitution was made silently, without any pause or comment, suggesting that it was part of an automatic solve-and-substitute-back sequence. When asked what he had done at that stage in the numerical example, he first recalled 'Once I'd worked it out, I put it back in'. This sentence is Frank's summary of the solve-and-substitute-back sequence. When prompted he remembered having substituted into $y = mx + c$. But he seemed not to notice that this time he had not substituted into $y = mx + c$ but into $q = mp + c$. The difference between substituting into $q = mp + c$ and $y = mx + c$ was not so keen to him as it was to me.

In an earlier conversation Eddie worked on the same problem. He also followed the sequence in a situation where it was not appropriate. He had obtained the equation $y = Mx + Mp - q$ and I asked him some further questions about it (the transcript extract is from 'Eddie' lines 166 to 171).

Liz Um, what I'm interested in is the coordinates of the point where it crosses the y -axis

Eddie Is it where .. just .. I'll divide through by M to get rid of the coefficient of x so it would be $M - p + \dots\dots\dots \frac{q}{M}$
so I take an M outside the bracket and just have $x - p + \frac{q}{M}$.. then
.... no, hang on, I'll just get rid of that last bit I've just done .. take the Mx over the other side, so you've got $q - Mp = -Mx$ then times through by negative one so you've got $Mp - q = Mx$. To make x the subject of the formula I'd then divide through by M so you'd have $p - \frac{q}{M}$ is the x coordinate

Liz Right

Eddie And I could also sub. that back in to find the y coordinate which would be $y = M(p - \frac{q}{M}) - Mp + q$ just multiply that out so you've got $y = Mp - q - Mp + q$, so the qs cancel out , so $y = ..$ well, zero, 'cos they cancel out as well.

In order to obtain the point where the line cuts the y -axis Eddie substituted $y = 0$ into the equation of the line (this mistake went unnoticed by both of us). Once he had arrived at an expression for x he followed the 'solve-and-substitute-back' sequence, using the phrase 'sub. that back in'. This substitution brings

him back to $y = 0$. The familiarity of the sequence drives him into this final substitution even though there is no need for it. (Later in the conversation he realises that this final substitution could be construed as a check on his working but it is apparent that that was not part of his purpose in making it).

Again at the end of the year Tommy was working on the same question, Problem D. He obtained $c = q - Mp$ and then said

'Therefore y is equal to Mp plus q take Mp (*writes $y = Mp + (q - Mp)$*)'

This is a hybrid between substitution into $y = mx + c$ and into $q = Mp + c$. I wanted to draw attention to the form of his equation $y = Mp + (q - Mp)$ and so I invited him to look at the form $y = mx + c$ and compare it with his answer (the transcript extract is from 'Tommy' lines 310 to 319).

Liz: Right. Have a look at this one again (*pointing to $y = Mp + (q - Mp)$*).

Tommy: There's no x in there. That should be x (*he changes the first p in $y = Mp + (q - Mp)$ to x*)

Liz: Ah hmm. Why do you need to change that to x ?

Tommy: Well we definitely consider p it will only be p , at this point here (*pointing to (p, q)*) But the general equation of a line, we've got to consider x .

Liz: Okay. Why do you think you got it wrong in the first place?

Tommy: Well I wasn't paying attention to it

At this stage in the year Tommy is quite articulate about why the equation of the straight line must contain x rather than p . He is also aware (after the event) that his attention was not on the equation into which he should substitute. The fluency of the solve-and-substitute-back sequence removed attention from the details of the substitution.

Each of these students demonstrates that the solve-and-substitute-back sequence is sufficiently familiar for them not to need to be prompted to use it, but that this familiarity makes it possible for them to use the sequence in inappropriate ways or circumstances.

In chapter eight I will show how an awareness of this sequence and other phenomena described in chapters five, six and seven contributed to my analysis of students' work on two particular tasks.

Use of 'forms'

In this section I am concerned with some of the 'forms' which play a role in the study of 'A' level maths.

$y = mx + c$ as the equation of a general straight line

$ax^2 + bx + c$ as a general quadratic

$y - y_1 = m(x - x_1)$ as the equation of a straight line with given gradient through a given point

These forms are so fundamental a part of 'A' level mathematics and so much a part of the teacher's mathematical experience that they are often introduced as though unproblematic. For example, at school A, in a lesson in October 1993 Peter started talking about the 'general quadratic' $ax^2 + bx + c$ without introducing it in any way. By contrast, a report of a project by the Mathematics Education Unit of the University of Auckland (Mathematics Education Unit 1994) includes a claim by one of the teachers working on the project that students became aware of the general form of a quadratic whilst involved in a calculator activity. At first the students, a class of mixed ability fifteen and sixteen year olds, solved quadratic equations by a trial and improvement method, using the calculator to compute for example $x^2 - 3x + 2$ for a number of values of x until they identified one for which the expression came to zero. Next they were introduced to the quadratic solutions mode on the calculator. This required them to enter into the calculator the appropriate information about a quadratic equation. The built-in programme required an input of the three coefficients. The teacher claims 'Because of their intimacy with the quadratic function in the guess and check activity it made sense to them that one of these types of functions could be identified from another by the coefficients a , b and c ' (p48). This was seen as a significant achievement for these pupils. I would identify this understanding as one of the first aspects of using a form. It is a formalisation, required in this case by the use of the calculator, of the visual recognition of a quadratic as being made up of three terms.

A little earlier in the lesson which I mentioned above, Lorne had replied to Peter's question, 'What is a quadratic?', by saying 'It's got three things in it - x^2 , x and $..$ '. His answer betrayed that his concept of a quadratic was predominantly a visual one. The 'formal definition' of a quadratic, as a polynomial in which the highest power of the variable is 2, stresses quite a different aspect of the concept. Lorne's description is of his visual summary of his experiences of quadratics to date. The 'formal definition' of a quadratic works by comparison with other polynomials which are likely to be experienced by the student at

some time after they have become familiar with quadratics. It relies on an understanding of the concept of 'polynomial' which is not available to students at a stage when their encounters with polynomials have been almost entirely with quadratics. So if the student is to gain an awareness of what distinguishes quadratics from other functions (or indeed of the wider notions of functions or polynomials) there is a role for the teacher in drawing attention to these features. The teacher's expert awareness must be made available, as much as possible, to the students.

The use of the form $ax^2 + bx + c$ stresses certain features of the concept of a quadratic and ignores others. Its use can thereby constrain and structure the way in which quadratics are conceptualised and used.

In this section I am concerned with the way in which these and other forms shape students' thinking and problem-solving strategies.

All of the forms which I consider in this section include x and/or y and one or more other literal symbols - a, b, c, x_1, m etc. In this section I shall refer to x and y as variables in this context and to other varying quantities as parameters.

Reliance on Forms in Problem-Solving

These forms, amongst many others are taught within a specific context in the expectation that they will then be available as a tool for use in unfamiliar problem situations.

To begin I will offer you an account of an incident which occurred in the course of my conversation with Robert, an 'A' level student. Robert's treatment of a mathematical problem triggers a recollection of my own recent learning of mathematics and I offer my own experiences as a parallel to his. You are invited to look for parallels in your own experience of teaching and of learning mathematics.

My attention in this incident focused on the form $y - y_0 = m(x - x_0)$ which Robert called on in his work on the problem. This form gives the equation of a straight line with gradient m and passing through the point (x_0, y_0) .

Robert and $y - y_0 = m(x - x_0)$

I asked Robert to work on some questions that I had found interesting. He became quite frustrated with the first problem because he couldn't remember all the details of the method he had chosen and it didn't seem to be getting him to the answer very quickly. However he was successful finally. I gave him a second problem as follows:

Problem R Find the equation of the tangent to the curve $y = x^2 + 1$ which passes through the origin

(the transcript extract is from 'Robert' lines 106 to 120)

Robert: Okay. Umm .. Oh tangents says derivatives straight away to me, $2x$ (he writes $\frac{dy}{dx} = 2x$) umm and so we do (writes) $y - y_0$ is equal to .. hmm .. I'm just going up here, it passes through the origin, I'm getting all mixed up here. .. Well it's going to be, m into $x - x_0$ Hmm what's going on here? Draw a little diagram here. We're expecting two because the .. graph's in here. We're expecting one there and one here. (he has drawn a diagram showing the curve and two tangents through the origin)

Liz: Umhmm.

Robert: .. Umm (he has written $y - b = m(x - a)$)

Liz: What's this point? (I am referring to the point (a, b))

Robert: Umm this is the point which is P here. (he marks the point of intersection of the curve and the tangent with positive gradient as P)

Liz: Right, okay.

(there is a short conversation where Robert tells me that he has tried a question like this before and was not able to do it)

Robert: well if I know that this point .. goes through the origin then surely y_0 and x_0 will be zero. Oh I don't know. I'm getting annoyed here.

Robert had no clear idea of how he was going to tackle this question. He was flustered because he had just struggled with another question which I gave him, which he and I thought he would find easy. He recognised that there was something in this question which was like a question he had found very difficult in the recent past.

So he began by differentiating the function, that is he began by performing a familiar algorithm which had been triggered by the word 'tangent'. Next he wrote down $y - y_0 = m(x - x_0)$, a form for the equation of a straight line which was again very familiar. He spent a lot of time in thought before substituting (a, b) for (x_0, y_0) and subsequently reconsidered whether he should have used $(0, 0)$ instead.

His attempts at the question were both channelled and constrained by his use of the 'form', $y - y_0 = m(x - x_0)$. The form demands the coordinates of a point which the line passes through. In this problem situation there are two points on the tangent which are of interest - the point of contact and the origin. Robert's focus on the form $y - y_0 = m(x - x_0)$ forced him to choose between them in an unhelpful way.

I am reminded of times when I have performed a familiar algorithm in order to see what will happen, or to give myself time to think, or to feel that I am doing something, but with no clear idea of what the outcome of my action will be.

I write down a standard form which I hope is relevant to the problem. When I have to decide what will replace the 'standard' parts of this form I realise that its application to this problem is not as transparent as I had hoped.

I use a formula which is familiar to me, but I don't feel in control of it. It controls me. I don't know what the outcome will be of the steps I am performing, or what I can do if it does not seem to present another way forward. The objects which I am manipulating are distant and shadowy.

In particular I remember a recent occasion when I was learning some 'mathematical methods' and found a problem which required the normal to a surface whose equation was expressed in the form $\phi(x, y, z) = 18$. I knew that one method of solution involved $\text{grad}\phi$ but I wasn't clear how. I calculated $\text{grad}\phi$. Then I looked up the solution which told me that $\text{grad}\phi$ itself was normal to the surface. I didn't understand why.

I can use the formula to produce $\text{grad}\phi$ but I don't know what else I can do with it. I don't feel comfortable with it. I can imagine there being lots of questions which I cannot answer about it. Its connections with other aspects of my mathematical knowledge are incomplete. My understanding is neither robust nor versatile. For me the form $\text{grad}\phi$ is a prompt rather than a complete script. It is a formula rather than a recipe, that is it specifies the ingredients but not the method of combination. It is an *aide-memoir* whose associated concept image (Tall and Vinner 1981) is only loosely and incompletely formed. It is a tool which I have used but not mastered. I write it down not because I have seen how it might be used but because I think it could be used somehow.

I find the notion of 'concept image' useful in describing my understanding of grad . My knowledge of grad is restricted to one type of use and so the potentially rich interconnections with other aspects of my mathematical knowledge have not yet formed. My concept image for grad is very sparse!

Robert's initial attempt at this question seems to me to be characterised by reaching for techniques and forms before having a strategy for solving the problem. This mode of working was also apparent in the next student's work.

On an occasion which I referred to in chapters five and six another student, Eddie, is working on what is to him an unfamiliar problem situation (the transcript extract is from 'Eddie' lines 45 to 113).

Eddie and the foot of the perpendicular

Problem J Find, in terms of a and b , the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$.

Eddie First of all I'd remember a few formulas,

Liz Okay

Eddie like the gradient of the normal line which is m and the perpendicular gradient is one over, negative one over m .

(Eddie appears to be using the word 'normal' mistakenly, i.e. not in its technical sense, that is to mean a line perpendicular to the tangent at a given point, but see notes below)

Liz Right

Eddie re-arranges the equation $x + 2y - 4 = 0$ to give $y = -\frac{1}{2}x + 2$. He says that the gradient of the 'normal' is then $-\frac{1}{2}$, and the gradient of the 'tangent' 2. He mentions 'the perpendicular' and I ask him what he means by that.

Eddie Er, well this perpendicular

Liz Do you know what the foot of the perpendicular means? Do you know what that term means?

Eddie No I don't to be honest

I explain to him the meaning of the term 'foot of the perpendicular'. He derives the equation of the perpendicular line and solves this equation simultaneously with $x + 2y - 4 = 0$, firstly for a and b , and secondly, at my suggestion, for x and y

Eddie .. so you could divide through by 5 which will give $y = \frac{1}{5}(8 + b - 2a)$

Liz Uhmhm

He substitutes his expression for y back into $x + 2y - 4 = 0$

Eddie ... so $x = \frac{4}{5} + \frac{2b}{5} - \frac{4a}{5}$ so you can take the $\frac{1}{5}$ out again so

$$x = \frac{1}{5}(4 + 2b - 4a)$$

Liz Okay, so what would you give as the answer to the question?

Eddie I'd give, er, 'in terms of a and b ...' Well I've found the coordinates, where these cross, in terms of x and y but you need to find it in terms of a and b .

Eddie's answers display that he was at best uncertain about the meanings of two important phrases in the question statement, that is 'the foot of the perpendicular' and 'in terms of a and b '. Yet he was willing to make a start by 'remembering a few formulas', rather than by planning a process of solution. He also worked quickly and confidently when he was performing algebraic manipulation, which he could do fluently without thought as to the meaning of the symbols with which he was working.

Because students habitually look for a form from which to start, these forms can become limiting of possible approaches. The student's attention then focuses on which particulars to substitute into the general form, rather than on relationships between the variables in the problem. For example Robert was constrained by his writing down of $y - y_0 = m(x - x_0)$ to search for the appropriate substitutions for x_0 and y_0 rather than attending to a wider view of the problem. He might have, for instance, seen that the point of contact was an unknown-to-be-found in this problem and conceive of the solution procedure as one of setting up equations to express the information given about this point. He later did see it this way. This approach to the problem was invisible at first because of his concentration on substituting into the form. Forms are so large a part of the teaching approach that they are seen as objects of teaching in their own right rather than tools which students must learn to use. The teacher may intend to teach the students *to use*, say, the quadratic formula, whilst the student is learning the formula *per se*. The form itself becomes the object of study, an item of content rather than a tool. So that a student, asked about whether a particular problem had been difficult, responded 'It's hard because you have to remember all the formulae'. On being given a problem to work on Eddie begins 'First of all I'd remember a few formulas'.

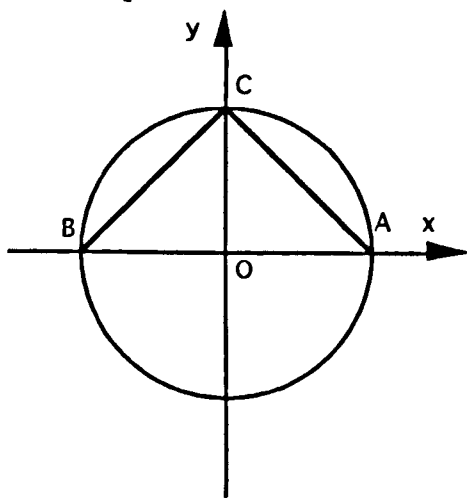
In a similar way to Eddie, I chose to calculate $\text{grad}\phi$ before thinking about what I would do with the answer. I hoped that the result would help me to see what to do next. Because the use of the form provided me with something to do, my attention was distracted from thinking through the relationship between the

function and a vector perpendicular to the surface. I was relying on metonymic connections to guide me through the problem, that is I hoped that the next stage would be triggered by the result of my calculation. Robert's reason for starting by differentiating was overtly metonymic ('Oh tangents says derivatives straight away to me'). I might contrast a more holistic view of the problem, one which would have allowed me to see the method of solution before starting, by describing this as metaphoric, in the sense of Sfard (1994).

I frequently invite a class to begin working on a problem by saying 'Who can tell me how we could start?', knowing that this kind of request is less threatening than 'Can you tell me how to solve this?' An alternative, which aims to stress metaphoric rather than metonymic thinking, might be to ask the class to work on producing an outline solution to the problem. In November 1993 I asked the students at school B to write outline solutions to some problems in coordinate geometry. Below I give an example of a student's response to this request. Further problems and the associated responses are given in Appendix H, 'Outline Solutions'.

Question: A circle, radius 2 and centre the origin, cuts the x -axis at A and B and cuts the positive y -axis at C. *Prove that $\angle ACB = 90^\circ$.*

Student's response:



Find the coordinates of A, B and C.

Find the lengths of AC and BC. Use Pythagoras' rule to see if $AB^2 = AC^2 + BC^2$. If it is, then $\angle ACB$ is 90° .

Some students needed to work through the question entirely before being able to write an outline solution of this kind. Some needed to go only part way through before doing so. Others could write the outline without attempting a solution. For most the task was a difficult one at first. I attribute this to their inexperience in speaking and writing about mathematics as well as the novelty

of the request to plan a solution without undertaking it. Later in the year I asked the students to undertake a similar piece of work. They were required to produce outline solutions to some exercises on trigonometry. This was very successfully completed without any apparent need to work through the problems first.

My aim was to encourage students to see the structure of the solution procedure rather than just the particular solution, that is, to concentrate on the stages to be performed rather than the arithmetic or algebraic manipulation required. This aim is in contrast to the experience I recognised from Robert's and Eddie's responses where actions are taken in response to metonymic triggers (or, explicitly in Eddie's case and perhaps implicitly in Robert's case, teacher suggestion). The student's thinking then becomes like a pin-ball, changing direction at every contact with a new suggestion.

I am sure that metonymic triggering also has an important role to play in problem-solving. It lies behind insights expressed as 'Oh, this is like the one I did before' and it is a means to achieve fluency. Similarly, asking students 'Who can tell me how we could start?' can be a useful beginning, encouraging them to contribute ideas even if they are ill-formed.

Rather than rejecting this as an approach to problem-solving, what I have done is to make a distinction between two types of approach, that is planning a strategy at the outset and responding to triggers or trying things out. The recognition of these two approaches suggests different ways of supporting others in problem-solving and has helped me to observe patterns of behaviour in myself and my students. Further I have identified an alternative action which I might take when I wish to avoid my habitual approach to problem-solving with a class of students.

The Notion of Form

In chapter 3 I mentioned the observations of Menghini (1994) on students' lack of awareness of form in algebraic manipulation. She also points out the difficulties students have with substituting expressions for letters, for example 'in analysis when $x + \Delta x$ is substituted for x ' (p12). Students of mine have, for example, suggested that if $f(x) = x^3$, then $f(x + \Delta x) = x^3 + \Delta x$. This seems to have something in common with Trevor and Paul's reluctance to substitute $b - 2a$ for c in $y = mx + c$. The replacement of a letter by an expression rather than a number is an alien idea and this is bound up with the difficulty of using letters as unknowns-to-be-taken-as-given (see chapter six). By the end of the year there is evidence that an enculturation has taken place for some students.

Earlier in this chapter, in the section 'Losing Track', I described my work on the 'affine transformations' task, Problem G. In that section I described how it led me to recognise 'losing track' as a phenomenon exemplified also in the work of Trevor and Paul. It also drew to my attention my own use of 'form' to find a way out of my confusion. Having acknowledged my difficulty I was able to use the question 'what form should it take?' to direct my thinking. I could use the notion of 'of the form' to decide how to manipulate the expression. It allowed me to see the expression as potentially 'something times the placeholder for x plus something'. I could separate the variable from the coefficient. My understanding of linear form encompassed substituting any expression independent of x in place of m and c in $y = mx + c$. It also encompassed substituting another variable name or expression for x . But even having used symbolic manipulation to organise the expression

$$(c + dm)x + dg + f = \frac{c + dm}{a + bm}[(a + bm)x + bg + e] + (dg + f) - \frac{c + dm}{a + bm}(bg + e)$$

into the form of a linear function of $(a + bm)x + bg + e$ I was not convinced by my argument. I had, finally, to call upon a much broader understanding of a linear function in order to satisfy myself that the result was justifiable. I had to draw on a 'concept image' of 'linear' which did not rely on visualising an algebraic form. (And incidentally the understanding of this result which was more satisfying to me comes from an argument which is much more difficult to convey).

My work on the affine transformation involved linear form i.e. the form $y = mx + c$. Rather than substituting into that form I was trying to recognise an expression as being of that form. In contrast to Trevor and Paul, my algebraic experience allowed me to admit quite complex expressions in place of each of m , x and c . That did not, however, remove the moments of doubt and bewilderment which I experienced on first being faced with the expression

$$\begin{pmatrix} ax + bmx + bg + e \\ cx + dmx + dg + f \end{pmatrix}$$

I therefore use the label 'of the form'. It allows me to contact both the usefulness of the notion of form of an algebraic expression or equation and also its complexity. A confidence in what can appropriately be substituted for the placeholders in a form (e.g. for y , m , x and c in $y = mx + c$) is no trivial matter.

Wenger (1987) identifies lack of awareness of form as a cause of difficulty for students. He gives an example of student difficulty in solving $v\sqrt{u} = 1 + 2v\sqrt{1 + u}$ for v (p219). Seeing it as a linear equation in v makes it easy. But typically students 'go round in circles' getting back to an equation they had already derived a few lines before.

The presence of the other variable, u , which is more usefully seen as a constant in the context of solving the equation for v , is a distraction. However it would, I suggest, not cause so many difficulties for students were it not accompanied by square root signs which trigger routines for isolating and removing them. I identify this as another example of the visual taking precedence over other aspects of form. The idea of a linear equation in v is not robust in the presence of strong visual distractions.

Boero (1993) gives an analysis of the process of transforming algebraic expressions in order to prove a conjecture or to solve a problem. He hypothesises that the two crucial ingredients are knowledge of standard patterns of transformation and anticipation of the next or final form. The above example from Wenger is perhaps an illustration of this point. Students who had mastered standard transformations of algebraic symbols without any strategic thought for the direction and purpose of these transformations would be likely to 'go round in circles' as described by Wenger. Students whose attention was on the final form of their answer, and therefore on the need to isolate v , would be more likely to succeed. Boero's point is that whereas schools (specifically Italian schools) place a lot of emphasis on standard algebraic transformations, they do not appear to spend very much time on the skill of anticipation.

Boero's analysis draws attention to the form of the next or final stage in the manipulation, whereas Wenger's example concentrates on the form of the equation given to solve. In other words one sees 'form' as a way of recognising the task, whilst the other sees it as way of planning the solution. Both uses of form are a way of guarding against the possibly unhelpful effects of metonymic triggering.

In Pozzi (1993a) the author describes a case study of a small number of sixth form students who were set tasks using Derive, the computer algebra system. The tasks encouraged students to 'discover' the formulae for differentiating products and quotients. The study aimed to observe generalisation, synthesis and abstraction in action and to examine how spontaneous or prompted use of notation might play a part. In order to see the form of $\frac{d}{dx}(u(x)v(x)) = u\frac{dv}{dx} + \frac{du}{dx}v$ students had to see the functions they were working with as generic representatives of general functions. This proved very difficult, with students instead developing intermediary 'generalisations', for example using x^n to stand for any polynomial.

The 'form' $\frac{d}{dx}(u(x)v(x)) = u\frac{dv}{dx} + \frac{du}{dx}v$ which the researchers wanted the students to 'see' is not always obvious in the final answer given by a computer algebra system. For example

$$\frac{d}{dx}(x\sqrt{(x^2 + 1)}) = \frac{2x^2 + 1}{\sqrt{(x^2 + 1)}}$$

does not easily betray a means of calculation to a student who does not already know it. So the researchers suggested functions which would give more 'readable' results, for example the product of a polynomial in x and either sine or cosine of x . Some students were able to reach a partial generalisation for this case, but it proved difficult for them to go further and see the polynomial and trigonometric functions as representatives of general functions. In effect the students were being asked to abstract from many particulars in an empirical generalisation. A more traditional approach to the teaching of this differentiation rule might ask students to experience a particular example as generic while the teacher directs the students' attention to what is generalisable.

The form, or formula, which is central to the object of this study, that is

$\frac{d}{dx}(u(x)v(x)) = u\frac{dv}{dx} + \frac{du}{dx}v$ is most commonly used to describe a process. It is rarely asked of students that they recognise an instantiation of this form, even in order to use it for integration. The task set up for these students asked them to see the sameness in result not in process.

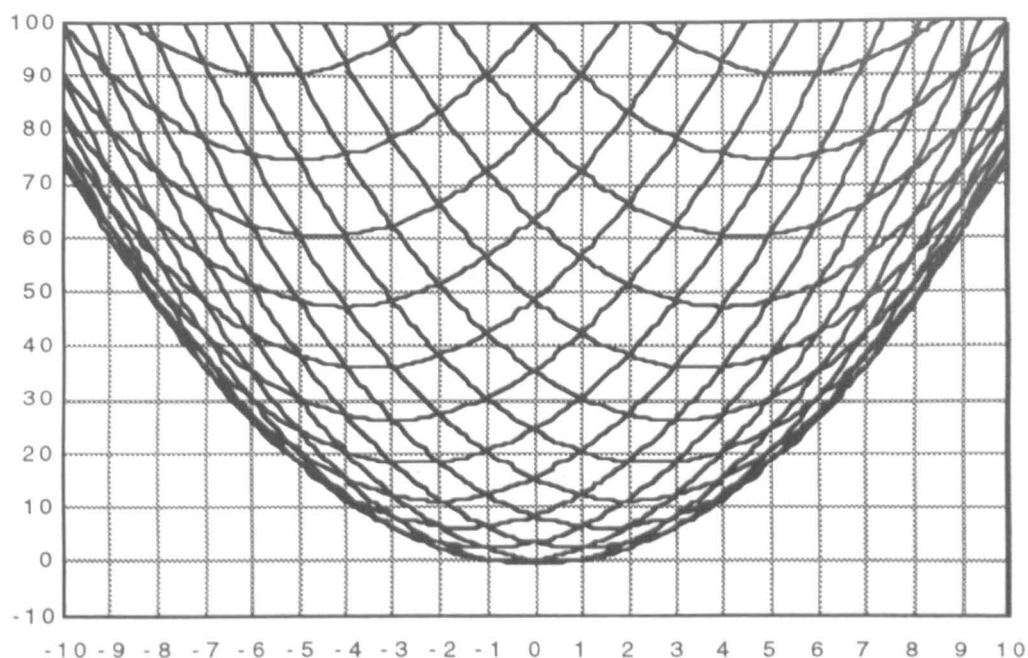
Envelopes task

My own work on another task again allowed me to draw parallels between what I experienced myself and what I read into my students' actions.

In order to demonstrate the facilities available on a graph plotting package I had programmed the package to plot a family of graphs for $y = x^2 + ax + a^2$ by varying the parameter a from -11 to 11 in steps of 1. The result is overleaf.

I was struck by the beauty of this image and by the extra curve which seemed to appear in it, apparently another parabola. I became more curious and found out that it can be referred to as the *envelope* of the other curves.

I asked myself a number of questions about such curves. How can I find the equation of this envelope? Would any other families of parabolae generate the same envelope? Could I have predicted that this set of curves would have an envelope, and if so that it would be a parabola? What kind of envelopes might be generated by families of other types of curves?



GraphPlot II ©1993 by Graham Cox

I thought through what would be the features of such a curve and how I might find the equation of this particular one. I used the idea of the envelope being tangent to each curve in the family, and an assumption that this curve was a parabola, to derive that the equation of this curve was $y = \frac{3}{4}x^2$.

A record of my work on this problem, which was written as I worked, is given in Appendix F, 'Envelopes'. Part of that account is given below.

First assume that the envelope is a curve which is tangent to each of the set of curves. That is that it has the same y -value and the same gradient at a point as each member of the generating set.

I will try to find the equation of the envelope in my diagram by assuming that it is a parabola. Because each of the set of generating curves is a parabola, the coordinates of the point at which each meets the envelope will be the solution of a quadratic equation. I can use my special knowledge of quadratic equations to say that two quadratic curves are tangent to each other if solving their equations simultaneously yields an equation with a repeated root. I am assuming only that the envelope is a parabola, so I will give it equation $y = px^2 + qx + r$. So I want to find values of p , q and r for which $y = px^2 + qx + r$ and $y = x^2 + ax + a^2$ have a repeated root when solved simultaneously for every value of a .

Put $px^2 + qx + r = x^2 + ax + a^2$

Then $x^2(p - 1) + x(q - a) + (r - a^2) = 0$

This equation must have equal roots for every value of a so

$$(q - a)^2 - 4(p - 1)(r - a^2) = 0$$

$$q^2 - 2aq + a^2 - 4(pr - pa^2 - r + a^2) = 0$$

$$a^2(1 + 4p - 4) - 2aq + (q^2 - 4pr + 4r) = 0$$

Since this equation must hold for all values of a , the coefficients of a^2 , of a and the constant term must each be zero:

$$1 + 4p - 4 = 0 \quad \Rightarrow \quad p = \frac{3}{4}$$

$$2aq = 0 \quad \Rightarrow \quad q = 0$$

$$q^2 - 4pr + 4r = 0 \quad \Rightarrow \quad r = 0$$

So the equation of the envelope is $y = \frac{3}{4}x^2$. This is confirmed by checking from the diagram that the envelope has its turning point at the origin and passes through (10, 75).

Is it possible to work back the other way i.e. to find the equations of the set of curves that would generate the parabola $y = \frac{3}{4}x^2$ as an envelope? Suppose I specify that I am looking for a set of parabolae of which $y = \frac{3}{4}x^2$ is the envelope. I need also to decide how I will represent that this is a family of curves. I think their coefficients must be a function of some parameter. Perhaps there could be more than one parameter. I will assume not to start with. So a member of the family that I am looking for will be:

$$y = f(a)x^2 + g(a)x + h(a)$$

I'm concerned that I will not be able to define so many arbitrary functions. I have given a form to this curve but I seem to have no information about it.

If this curve touches the curve $y = \frac{3}{4}x^2$ for each value of a then:

$$f(a)x^2 + g(a)x + h(a) = \frac{3}{4}x^2$$

has repeated roots for all values of a .

$$x^2(f(a) - \frac{3}{4}) + g(a)x + h(a) = 0$$

has repeated roots if

$$(g(a))^2 - 4h(a)\left[f(a) - \frac{3}{4}\right] = 0$$

Since this must be true for all a , is it true that each term must be zero i.e. that $g(a)$ must be zero and that either $h(a)$ or $f(a) - \frac{3}{4}$ must be zero? No, because I am

dealing with functions, not constants. To say that $g(a)$ must be zero is to say that it must be the zero function. In fact I cannot simplify the statement any more than I have done. If I decide on functions for $f(a)$ and $h(a)$, then $g(a)$ is determined.

Although I was writing this account as I worked on the problem there was an inevitable time lag between thinking and writing. Hence my dilemma over whether each term should be zero was resolved before I expressed it in writing. What I have written in the account does not adequately capture the experience of that dilemma. I was aware that I had introduced three unknown functions, which have a feeling of being much more undefined than unknown numbers. Moreover I had only one equation from which to obtain information about these functions. I was anxious to get as much information as possible from that one equation. The process of setting each coefficient to zero, which I had used in the previous calculation and which had the effect of generating three equations from one, was fresh in my mind. I was also aware that the equation I had formed must be true for all values of a , which had been my justification for setting the coefficients to zero before.

My first instinct was to use the same procedure again. But my concern that I was dealing with functions and not single values made me pause before doing so. I decided that setting each term equal to zero was not applicable because the unknowns were functions and not constants.

Looking back on this work now, I justify not setting each term equal to zero differently. The process of equating coefficients to zero is valid when a polynomial is identically zero. However the equation I was considering applying this process to was not based on a polynomial. I was contemplating setting each *term* equal to zero and neglecting the difference between a term and a coefficient.

Once I had decided that the process of equating coefficients to zero was not appropriate I could recognise that the equation I had, and which I was trying to decide what I could do with, was in fact my 'answer'. It was 'as far as I could go'.

Two aspects of my dilemma here strike me as parallel to students' difficulties.

First, in the moment I was not attending to what I see in retrospect as the significant aspect of the equation, that is that it was not a polynomial equation. The comparisons which I was making between it and my previous calculation concentrated on the similarities (that there were three unknown objects, that I required as much information as possible from one equation) and on the one

difference, namely that the unknown objects were functions and not quantities. The superficial similarities between the two cases persuaded me to consider treating them similarly. I had missed the difference which was most significant with respect to what I would do with the equation, that is that it was not a polynomial equation.

The parallel I draw here is with Frank's behaviour in failing to distinguish between $y = mx + c$ and $q = mp + c$. The superficial similarities distracted his attention from the significant differences, in this case between the variable nature of x and y , compared with the 'variable-constant' nature of p and q . The visual form of the equation took precedence over the structural form.

Secondly, I did not at first realise that the equation I had in front of me was 'as far as I could go'. I was still looking for an 'answer'. I had not identified for myself what sort of answer I was expecting and I did not recognise this equation as giving me any immediately useful information. I unconsciously hoped for something more concrete.

In a similar way both Frank, and Trevor and Paul (see 'Unknown-to-be-taken-as-given' in chapter six), obtain an equation for c (in Frank's case $c = q - mp$ and in Trevor and Paul's case $c = b - 2a$) but do not recognise it as an 'answer'. They are still looking for some more concrete form for c .

In each case my recognition of parallels in my own mathematical experience allows me to form a further understanding of (in the sense of metaphor for) the students' struggles with these problems.

Substitution into forms

Within coordinate geometry and the algebra of functions it is possible to describe consistently some roles as those of variables and some as those of parameters. The variable roles are consistently played by x and y and the parameter roles by other letters. Whilst any letter may change its role during the course of a problem solution, in particular to that of unknown, the roles played within the 'standard forms' of these areas ($y = mx + c$, $ax^2 + bx + c$ etc.) can be meaningfully described as variable or parameter.

Some processes involving these forms require students to substitute numbers for parameters to obtain a particular instantiation of the form. For example the task 'Write down the equation of a cubic curve which cuts the x -axis at $(1, 0)$, $(2, 0)$ and $(3, 0)$ ' might be approached by substituting 1, 2, and 3 into the form $y = a(x - b)(x - c)(x - d)$. This kind of substitution might be conceived of as a move from general to particular.

Some processes involving forms require the substitution of numbers for variables and a further use of the result. For example, the task 'Write down the equation of a cubic curve which cuts the x -axis at $(1, 0)$, $(2, 0)$ and $(3, 0)$ and passes through the point $(0, 6)$ ', might be approached by substituting 0 and 6 for x and y in the form $y = a(x - 1)(x - 2)(x - 3)$ in order to arrive at a value for a . Using this value for a would mean substituting it for a back in the form again. This kind of process involves the 'solve and substitute back' sequence.

Some processes involving forms require a substitution of expressions, rather than numbers, for parameters and/or for variables. These expressions include letters which are taking the role of 'unknown-to-be-taken-as-given' as I described it in chapter six.

Below, a student, Paul, demonstrates that replacing parameters by numbers in the form $y = mx + c$ is familiar ground for him but that dealing with symbolic expressions in place of m and c is more challenging.

Paul and comparing a form to $y = mx + c$

I asked Paul to work on

Problem D Find the equation of a straight line which has gradient M and passes through the point (p, q)

(the transcript extract is from 'Paul' lines 14 to 46)

Paul: Oh right. Okay. First of all if I remember correctly the formula is something like that (*writes $y - y_1 = m(x - x_1)$*) or there is a formula of something like that (*writes $y = mx + c$*), yes? Which then go to, (*writes $y - q = m(x - p)$*)..... like so, substituting p and q for y_1 and x_1 - then I could rearrange that, multiply that out first. (*writes $y - q = mx - mp$*) Put it all on one side over there (*writes $mx - mp - y + q = 0$*) The only problem is that gives me .. a lot of letters - but not (*inaudible*). It's not in a form where I've got $y =$, oh I don't know. Usually I get a ' $y =$ ' form. Usually a y over there. Umm I'm trying to get $y = mx + c$. So you've got $y = mx$, there's the mx there (*writes $y = mx - mp + q$*). So that (*pointing to $-mp + q$*) is going to equal c but I don't know how. The p add the gradient times the two coordinates. No, gradient times the x coordinate add the y coordinate. I'm not sure how

Liz: So what does that bit represent? (*pointing to $-mp + q$*)

Paul: This bit here?

Liz: Hmm.

Paul: This represents the point where it crosses the origin, is that right?. The c , the c part - so that bit will equal c when $p = 0$, so it's going to cross the y -axis, it's going to cross the y - oh..... because $mp + q$ except it's passing through a point, isn't it, so it's not going to be (*inaudible*)

Liz: What are you doing?

Paul: It's, I was assuming that p was the x -axis as opposed to a point. An x coordinate - whereas actually it would be where $x = 0$ that that equals c . So that bit there (*pointing to $-mp + q$*), is the point where it crosses the y -axis, but I'm not quite sure why, why that should be - perhaps it just is

Liz: Can you do me a sketch of $y = mx + c$.

Paul: $y = mx + c$. It's just going to be a line. Could be any line

Liz: Well, umm, okay select an m and c and do it.

Paul: This is the cheat's way out - where m was 1 and c was 0. (*draws a sketch of $y = x$*)

Liz: Okay.

Paul: $m = 1$, $c = 0$ (*reading this out as he writes it alongside his sketch*) If I just keep, if I change, if I change this c value, it just, and keep the gradient the same and just move the line up and down, the axis so $m = 1$ and $c = 1$, just move up like that across the point. (*draws a sketch of $y = x + 1$*) Whereas if I change the gradient from the original one it - it's just basically going to get steeper (*draws $y = 2x$ and writes beside it $m = 2$, $c = 0$*) or shallower like that (*draws $y = \frac{1}{2}x$ and writes $m = .5$ $c = 0$*)

In this last speech Paul demonstrates his familiarity with the meaning of m and c in the form $y = mx + c$. Earlier on he has replaced x_1 and y_1 in $y - y_1 = m(x - x_1)$ by p and q but he is then not sure of the status of the result. He is not happy that he has finished, saying 'I'm trying to get $y = mx + c$.' and also 'The only problem is that gives me .. a lot of letters'.

I wish to draw a distinction between, on the one hand, using the form to name or to summarise, and, on the other hand, manipulating it. By naming I mean, for example, using $y = 3x + 1$ to name a line with gradient 3 and y -intercept 1, by reference to the general form $y = mx + c$. By summarising I mean understanding $y = mx + c$ as a summary of the family of straight lines.

Manipulating, on the other hand, involves substituting expressions for one or more of the literal symbols in the form (i.e. for x , y , m and/or c) and maintaining the naming and summarising function of the form.

Substituting for the parameters, m and c , in the form is part of the naming and summarising use, whereas substituting for the variables x and y in the form requires a different understanding of the form and will lead, if the substituted values are expressions rather than numbers, to a manipulative use of the form.

In these terms Paul is able to use $y = mx + c$ to name and summarise but is more hesitant over manipulating it.

His confidence in using the form $y = mx + c$ is reduced when the particulars which replace m and c are expressions rather than numbers. The presence of p and q in the expression $-mp + q$ is a distraction as Paul recognises when he says 'I was assuming that p was the x -axis ... as opposed to a point'.

Summary

Sfard and Linchevski (1994) (p196) in speaking of the operational phase of algebra as a historical period give this problem and solution as an illustration of the characteristics of the period.

from Al-Khwarizmi (after Struik 1986 p58)

The problem: What is the square which combined with ten of its roots will give a sum total of 39?

Solution: take one half of roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself give 25, an amount which you add to 39, giving 64. Having taken the square root of this which is 8, subtract from it half of the roots, 5, leaving 3. The number 3 therefore represents one root of this square, which itself, of course, is 9.

If you have not already done so, try to understand the process which is being used to solve this problem.

I was particularly struck by my reaction to this problem and the given solution. I read it through several times and eventually overcame my difficulty in understanding the references to a 'square' and 'its roots'. In my mind these phrases summoned up the notation x^2 and \sqrt{x} , so that the latter was not the root of the former, but of the unknown. Once I had accommodated to think of the root as x , the solution was easier to follow and I soon 'recognised it' as 'completing the square'. This recognition was accompanied by an element of surprise because my mental image of completing the square is full of x s and brackets, both of which are conspicuously absent from this account.

The symbols which were part of my education in algebra had also become part of my visual image of the process of completing the square. Having recognised this solution as being 'the same as' completing the square I was no longer interested in it because I had comprehended it in terms of my pre-existing knowledge schemas.

Forms can give a feeling of security and a means of accommodating new situations into existing knowledge patterns. Visual aspects of form are very important in recognition and in mental manipulation.

Substitution is at the heart of the meaning and use of the forms I have been discussing. Distinguishing between substitution of numbers for parameters, numbers for variables, algebraic expressions for parameters and algebraic expressions for variables allows me to recognise the different awarenesses needed to operate at each of these levels. I have linked these different kinds of substitution with naming, summarising and manipulating the form in which the substitution takes place.

Substitution is so prevalent a procedure in using these forms that it can become habitual and not available to conscious thought, as in the solve-and-substitute-back sequence. The meanings of such substitutions can be very subtle, as the roles of variables change during the solution process.

As in my own case, when working on the Envelopes task, the superficial appearance of an equation took precedence over its more structural characteristics, so forms which are familiar to students may have more visual than formal familiarity.

In addition, as in the case of Frank (see chapter five 'Seeing the General in the Particular'), a general form may not be recognised as a summary of particulars. In this case the form is not informing because its relevance to particular situations is not seen.

In the same way it is possible to acknowledge general principles concerning the teaching and learning of mathematics without becoming aware of their use in particular situations, that is to see the general form but not the particulars it aims to summarise. It has been my intention to use many particular instances in this thesis in order to bring awarenesses to expression in a way which will inform practice, rather than remaining as empty forms.

My next chapter begins by addressing more specifically the question as to how some of these awarenesses might be brought to bear on a particular situation which arose in my practice.

Chapter 8 Analysis of Two Problems

In this chapter I give several accounts of individuals' working on two different tasks. I have chosen these two tasks because I have found that my analysis of the work done by students and others on these tasks can be informed by consideration of a great many of the issues which I have raised in chapters five to seven. Part of my purpose in this chapter is to demonstrate how the awarenesses which I have developed in the earlier chapters might be used to inform practice.

The first account of students' work ('Deriving $y - y_1 = m(x - x_1)$ ') is followed by a number of possible interpretations of the students' actions. These are based on the themes I have developed in chapters five, six and seven. Each story is accompanied by a description of a teaching gambit which aims to address the aspects of the students' awareness which I identify. Many of these have proved successful in my classroom, in that they have directed students' attention to the points on which I wished them to focus. You are again invited to review your own teaching experience for resonance with or reaction against what I have to say.

The second set of accounts ('Tangential') constitutes a demonstration of the potential richness of a single task to offer awarenesses concerning the roles of variables.

A third aspect of my purpose in this chapter is to add weight to the assertion in my Introduction that understanding the roles of variables is a key component in working successfully in second variable situations.

Deriving $y - y_1 = m(x - x_1)$

The following is a description from my diary of a lesson from November 1993:

Quite early in the lesson I ask them to find the equation of a straight line with gradient m and going through the point (x_1, y_1) . Although we have done two examples of this process using numerical values for m , x_1 and y_1 and they have described the process to each other, they find this very difficult. For example, having used the equation $y_1 = mx_1 + c$ to find an expression for c , Hal substitutes this expression into the equation $y_1 = mx_1 + c$ and gets $0 = 0$. Tommy writes the equation as $y \equiv mx + y - mx$. I ask him whether some of the x s and y s should have subscripts. He thinks for a while and then adds subscripts to all of them. There is again some difficulty when I later ask them to check their formula with a numerical example. Lorne begins by substituting the values 2 and 8 (the given point is $(2, 8)$) for x and y rather than x_1 and y_1 .

The method which had been developed in this class for finding the equation of a straight line with given gradient through a given point is as follows.

First substitute the given gradient for m in $y = mx + c$. Next replace x and y by the coordinates of the given point in order to calculate a value for c . Then write out the form $y = mx + c$ with the appropriate substitutions made for m and c .

Before setting the task described above to the class I had spent some time developing the idea that any line with gradient 3 could be represented by the equation $y = 3x + c$. I then demonstrated a method for finding the equation of a straight line with gradient 3 that passes through the point $(2, 8)$. Next I asked the students to find the equations of two more straight lines, given their gradients and one point which each passed through. Finally I asked the students to describe to each other in general terms what was the method which they had been using.

There are a number of stories I can use to explain the events I have described from this lesson and I list them below. In these stories I use the terms and language which has been developed through chapters five, six and seven. Each story is linked with a teacher action designed to enable pupils to work on the awarenesses identified in the story. As the reader you will almost inevitably test my interpretations against your own experience in any or all of a number of ways. First you may recall students working on similar tasks and decide whether or not my descriptions fit the actions of your students. Secondly you may think about whether the awarenesses I describe are central to the task the

students were attempting. Thirdly you may consider whether my explanations of the students' behaviour seem plausible. Fourthly you may, as a thought experiment or in the classroom, try out the teacher actions I suggest.

The stories are not intended to be mutually exclusive or to act as alternatives to each other. All or none may be consonant with the experience of a particular individual.

Story 1

Hal set out on a procedure which he had practised several times and had rehearsed by describing it to a neighbour. Replacement of m and c by numbers to produce the equation of a particular straight line was familiar and comfortable. But here x and y must be replaced by other literal symbols. The equation resulting from this substitution did not apparently yield a 'value' for c , as it would in numerical examples, because it contained other literal symbols. The departure from the anticipated pattern (the appearance of a value for c) created an uncertainty as to the next step. This apparent failure of the previous method caused Hal to 'lose track' of the procedure he was following. My use of this label follows my discussion under the heading 'Losing Track' in chapter seven.

Teacher action *Ask students to describe to each other in general the method they are using. Ask about the purpose of each step.*

One possible approach to this difficulty is to give students opportunities to make their solution procedures more robust against the distractions of slight changes to the form of the task. The requirement to describe to another student the solution procedure in general terms with an explanation of the purpose of each stage could provide such an opportunity. I was aware of this strategy at the time of teaching this lesson and had set a similar task to the students. However, I was aware that most students were in fact describing the stages of the process in terms of one of the examples, rather than in general.

Story 2

In deriving the general form $y = mx - mx_1 + y_1$ students must replace c in $y = mx + c$ by literal symbols or an expression involving literal symbols, rather than by a number. In working on particular cases there was a clear distinction between x and y which in the final equation remained as letters, and m and c which were replaced by numbers. In other words there was a distinction between the variables in the form and the parameters. The final equation could be seen to 'look like' the equation of a straight line in a purely visual

sense and conformed to the students' previous experience of the equations of straight line graphs. I use the word 'visual' as a qualifier because the phrase 'looks like' is commonly used by mathematicians to mean 'conforms to a form', and my meaning is different from this. It is possible to recognise, for example, $y = 3x + 4$ as the equation of a straight line in a visually impressionistic way, without checking that it conforms to the pattern 'y equals some constant times x plus some constant'. (I have expressed this form in words here, rather than symbols, to stress its non-visual sense). If, then, the student's concept of the equation of a straight line is based on a visual image, a final form which includes expressions instead of numbers as coefficients will not fit this concept. Working with expressions not numbers blurs the distinction between parameter, which is replaced by a number, and variable, which stays as it was. It also disturbs the expected visual form of the 'answer'.

Teacher action Discuss which of these are the equations of straight lines:

$$y = Ax + b$$

$$s = 2t + 5$$

$$y = \frac{1}{2}x^2$$

$$y_1 = mx_1 + c$$

$$y = m^2x + c$$

Which are equations of *general* straight lines?

Ask students to describe in words what the equation of a straight line is like and to anticipate what form their answer is going to take.

Directing attention to form in this way might help students shift their attention away from the visual aspects of the form of the equation and towards a more formal conception. The requirement to describe in words what characterises the equation of a straight line forces awareness of existing visual representations.

Story 3

I have described this process of obtaining a value or expression and substituting it back into a form as a solve-and-substitute-back sequence in chapter seven. Having achieved a 'value' for c , students must decide what to do with it. The solve-and-substitute-back sequence dictates that the expression must be substituted somewhere. Looking for somewhere to substitute it back they tend to choose the equation $y_1 = mx_1 + c$ from which they have just derived their expression, rather than choosing $y = mx + c$ itself. In glancing back up the page for some equation to substitute into, their eyes rest first on $y_1 = mx_1 + c$ because it was written most recently. In fact $y = mx + c$ may not

have been written down at all. For some students this solve-and-substitute-back process became automatic. By this I mean that 'substitute back' was so strongly suggested to them by the action of solving to get an expression for some literal symbol, that there was then no opportunity to question whether and where the substitution should be made.

Teacher action When students are working on such a problem watch for the moment at which the substitution is made and challenge them to account for their action.

One way of trying to break into the automaticity of the solve-and-substitute-back process is to enable students to recognise the state of being caught in it. This might be best achieved by recognising the moment of automatic response when students are working and intervening to direct their attention to it. This relies on the teacher being in the right place at the right time. Perhaps a more realistic alternative is to draw attention to the point at which I decide where to substitute back as I am working through a similar problem in front of the class. It might also be possible to prompt students to re-enter the moment after the event. The recognition of the drive to substitute back might become a trigger in future to consider where the substitution should be made.

Story 4

The difference between the two equations $y_1 = mx_1 + c$ and $y = mx + c$ is insignificant for the students because the m and c are the salient features of each. In a sense the x and y are 'invisible'. In $y = mx + c$ the students pay little attention to x and y . They are taken for granted because they are always used. I have drawn attention to this phenomenon under the heading 'What is special about x and y ?' in chapter six. It is m and c which are noticed because attention in the past has been on replacing them by numerical values. This effect is compounded by the use of x_1 and y_1 as particular values of x and y because the equations are then superficially similar. Subscripts are frequently experienced by students as being confusing and may therefore be suppressed by them to leave x , y and x_1 , y_1 indistinguishable.

Teacher action Ask for the students' help in writing the question - what should we use as the coordinates of the general point through which the line is to pass? Why is (x_1, y_1) frequently the choice of text book writers?

In drawing attention to the arbitrariness of the choice of (x_1, y_1) it is possible to distinguish it from the conventionally compulsory use of (x, y) . This can be used to highlight the difference between the two pairs and the role of the subscripts in this difference.

The wording of this question 'What should we use as the coordinates of the general point?' is significant. Alternatives which superficially ask the same are 'Give me a general point on this line', 'What shall we call this general point?' My original wording stresses that the choice of the point comes before the derivation of the equation of the line. We are not looking for a point on a line, but for a line through a point. The label $(x, mx + c)$ is a suitable answer to either of my rewordings of the question, since it represents a point whose coordinates satisfy the equation of the line. But it is not a suitable answer to the original question since no relationship has yet been assumed between the coordinates of the point.

Story 5

The second form which is involved in this episode is $y - y_1 = m(x - x_1)$ or its equivalent $y = mx + y_1 - mx_1$. This is the equation which I hoped the students would derive. All the students eventually had one of these equations or another equivalent derived and written down in front of them. I asked them to check their 'formulas' by using them to find the equation of a line with gradient 3 passing through the point (2, 8). We had worked this out as $y = 3x + 2$ by a different method earlier in the lesson. The majority of the students did not see how their equations could be used to find the equation of this line. In other words they did not link their equations with the writing down of a particular line with given gradient through a given point. The equations they had derived had not yet become forms for them. They did not see them as a general summary of many particulars. They did not see the particular in the general in the sense which I have outlined in 'Seeing the Particular in the General' in chapter five.

Teacher action Ask students to consider the meaning of each literal symbol in their 'answer' and to determine its role. Which are to be replaced by a particular and which are to remain as they are? Which are 'constant' and which are variable?

Ask students to formulate a question which could most efficiently be answered by using their formula.

Appreciating the formula I have just produced as a general form requires a shift in attention from one aspect of the roles of x_1 and y_1 to another.

These letters represent the coordinates of a particular point and at the same time a general point. In this sense their roles are those of parameters. By the end of the procedure they have taken a place in the form

$y - y_1 = m(x - x_1)$ where their role is unknown-to-be-given. When the form is used in further problems they will be replaced by particular

numbers or expressions. The shift in their role is very subtle. It is a shift from seeing the point (x_1, y_1) as a general point to seeing it as an unknown-to-be-given. Paying some attention to the transition from deriving the formula to using it may help students to make that shift. This shift is also typical of many situations in 'A' level mathematics. Whenever a general form is derived and then used in particular cases such a shift takes place.

Each of the stories I have used to explain the students' actions is based on my observations as described in earlier chapters and has applications beyond this particular problem. I contend, then, that the stories and teaching gambits that I have put forward as a response to my students' work on this task address fundamental awarenesses for this area of mathematics.

Tangential - Expressing Particular and General

Below are accounts of how some students and teachers tackled two related problems in which the complexity of the roles of variables within the situation was compounded by the need to make a choice of letters with which to express a generality. Again I have used the ideas developed in chapters five to seven in my analysis of the work done on this problem.

Both groups of teachers with whom I met also worked on this problem. My reason for including it here, and for gathering all my accounts of work on this task together rather than distributing them by theme in earlier chapters, is that this task, above all the others which have been mentioned in this thesis, proved to be very rich in provoking awareness and discussion amongst the teachers and in producing a wide variety of interesting responses from students. In using my accounts to demonstrate this richness, I present an argument for the usefulness of teachers' working on mathematical tasks as part of their personal professional development.

You will gain more from reading what follows if you have worked on the problems for yourself before reading about the students' solutions.

The original problem was as follows:

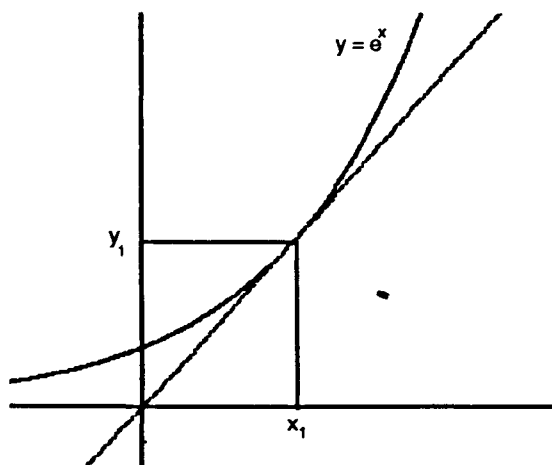
Problem S Find the points on the curve $y = e^x$ at which the tangent goes through the origin. Find the points on the curve $y = e^{2x}$ at which the tangent goes through the origin. Repeat for $y = e^{3x}$, $y = e^{3.5x}$, and generalise. The set of points you found appear to form a 'curve'. What curve is it, and why?

A group of students worked on this task, which was entitled 'Tangential', at a sixth form mathematics day. At the end of the session they were asked to

produce a poster to display their work. I have reproduced their poster as accurately as possible below:

TANGENTIAL

$$y = e^x$$



gradient of
tangent = e^{x_1}

$$\begin{aligned} \text{equation of tangent} &\Rightarrow y = e^{x_1}x \\ &\Rightarrow y = e^x. \end{aligned}$$

$$\therefore e^{x_1}x = e^x.$$

$$\therefore x = 1 \Rightarrow y = e.$$

$$y = e^{2x}$$

$$\frac{dy}{dx} = 2e^{2x} \quad \text{equation of tangent } y = 2e^{2x}x$$

$$\therefore e^{2x} = 2e^{2x}x$$

$$\therefore 1 = 2x \Rightarrow x = \frac{1}{2} \Rightarrow y = e.$$

GENERALLY:-

$$y = e^{nx}$$

$$\frac{dy}{dx} = ne^{nx} \therefore \text{equation of tangent } y = ne^{nx}x$$

$$\therefore e^{nx} = ne^{nx}x \Rightarrow \frac{1}{n} = x$$

$$\Rightarrow y = e.$$

Graph of coordinates of the points where the tangents that pass through (0, 0) touch the curve.

On a first reading I was struck by the fact of the various meanings of x in their workings. For the first curve they drew a sketch. In their diagram they recognised the *particular* nature of the point of tangency by labelling it (x_1, y_1)

and the gradient of the tangent e^{x_1} . However, in their working to find the x coordinate of the point of tangency, they dropped the subscripts and wrote $y = e^x x$ as 'equation of tangent'. According to one analysis, the first ' x ' in this equation refers to a particular value of x which occurs at the point where the tangent touches the curve. The second ' x ' is a variable in a relationship between x and y which can be represented by a straight line.

At this stage, in their notation they have lost the distinction between x as a variable in an expression of a relationship represented by a straight line (e.g. in the equation $y = mx + c$), and x as a particular-but-as-yet-unknown coordinate of the point of tangency (as expressed by x_1). Both these meanings of x exist within the same equation. In Adda's terms (Adda 1982) they have introduced a homonymy into the equation by using the same symbol to stand for different quantities.

If I try to interpret the equation $y = e^x x$ without assuming an ambiguity in their use of notation I have two options. I can see x and y as variables connected by this equation, which could be graphically represented by a curve (not a straight line). Alternatively I can see it as a statement which is true at only one point on the original curve and only one point on the tangent, that is at the point of tangency. In neither of these cases can I see it as the equation of a tangent.

In the next equation, $e^x x = e^x$, they equated the y -coordinate of a point on the tangent with the y -coordinate of a point on the curve in order to find the value of x which allowed this equality. Now each x in the equation refers to the particular-but-as-yet-unknown value previously called x_1 and x is unambiguously an unknown-to-be-found.

The elision between the two meanings of x did not prevent the students from going on to find the particular value of x required. However this group nowhere recorded the equations of the tangents whose points of tangency they were interested in. It is possible that they chose not to do so because it was not required for their further investigation and not suggested in the problem outline they had been given. However I suggest that it is common for students who have consciously set up and solved an equation to find the value of a particular-but-as-yet-unknown to announce the fact of having found it by replacing the 'placeholder' that they have used by the value they have just located. I suggest further that not doing so may be a sign that ~~they~~ they have not seen the process they have just undertaken as one of finding a value which satisfies certain conditions.

It seems that I can at least say that the distinction which I make between variable and unknown was not as sharp for the students as it is for me. The

students did not see the need to use two different letters to make the distinction. This behaviour may signal an elision of meaning between variable and unknown. I draw a parallel between this and my colleague's students who did not see the need for a second letter in their proof of divisibility by nine, and also between this and the summer school students who did not see a need for a second pair of letters to express a second element of the group (see p119 for both of these accounts). Choices made in the use of letters to express relationships in the problem situation (for example, here the choice not to use x_1 in the expression of the equation of the tangent, for my colleagues students the choice to use n in both cases) give some information about the way in which those relationships are seen by the learner.

Teachers working on Tangential

Both groups of teachers with whom I met regularly worked on this problem and considered the students' solution.

In one group Lesley worked on her own. She differentiated e^x , wrote $y = mx$ as the equation of the tangent and labelled the unknown point as (a, e^a) . Then she wrote $e^a = ae^a$ and solved it to get $a = 1$.

The others in the group were still working. They had called the point of tangency (x_1, y_1) and were trying to use the fact of intersection of tangent and curve, but without having differentiated, to find the gradient of the tangent.

Instead they were using $\frac{y_1}{x_1}$ as the gradient.

Lesley intervened by saying

Lesley I may have got it all wrong - if it goes through the origin the equation of the line is just $y = mx$

Others yeah - we got that

Lesley the gradient of the curve - the gradient of the line is e^x

Others whoa - what? - the gradient?

Lesley - yeah because when you differentiate e^x you get e^x yeah? - so the equation of the line is $y = e^x x$

Others oh my god! - you gave this to kids?

$$\text{so } \frac{dy}{dx} = m$$

so what have we got? - so $y = mx$ - what do you do with that then?

Hang on - no it's $\frac{1}{x}$ isn't it? What's the differential of e^x ?

e^x - so y is equal to $e^x x$ - so if m is e^x

Lesley So now all you have got to do is to consider the point - what happens at a particular point

Others $y = e^x x$

Later in the session I asked the teachers to consider generalising their solution to this problem and Lesley did so by replacing $y = e^x$ by $y = f(x)$. She went on to apply it to the particular case of $y = \ln x$. She used x as the x coordinate of the point of contact, rather than a as she had in the first problem, and so at the stage of writing down the equation of the tangent she got $y = \frac{1}{x}x$. She decided to change this by using the notation she had developed in the first part of the question.

Lesley showed her awareness of the difference between the roles of variable and unknown-particular in her initial use of notation. However when she advised the others in the group about finding the equation of the tangent she used x in an ambiguous way. She then continued with this use in her work on $y = \ln x$. On seeing $y = \frac{1}{x}x$ as the equation of the tangent she realised that there was a difficulty with using x in this way. The equation $y = \frac{1}{x}x$, because it invites cancellation to leave $y = 1$, made this difficulty visible in a way which neither $y = e^x x$ nor $y = xf'(x)$ did.

This ambiguous use of x occurs because of a shift from variable in the equation of a curve (in the equation $\frac{dy}{dx} = e^x$, x stands for the first coordinate of any point on the curve $y = e^x$) to unknown particular value of x (in the statement 'gradient of tangent = e^x ' x stands for the first coordinate of the point where the tangent touches the curve). A further complication arises because of the use of x as the first coordinate of any point on the tangent.

This first shift, from variable to unknown, was marked by some of those who worked on this question by a change of notation. The second meaning, of unknown particular, was signified by a different letter. This kind of shift, and its reverse from particular to variable, are very common in A-level mathematics. It occurs, for instance, when I try to locate a particular point on a given curve which satisfies certain conditions; whenever the point of intersection of two curves is found by solving their equations simultaneously. I

describe my experience of a shift in the reverse direction in the section headed 'Particular becomes General' in chapter six.

In some cases this shift can be thought of as a change in the domain of validity of the equation. For example, in the 'Tangential' question the equation $\frac{dy}{dx} = e^x$ is first conceived of as true at any point on the curve $y = e^x$ and later applied to just one point on the curve, i.e. the point where the tangent touches. One approach to these situations is to use a different notation to mark the shift. The notation change might be to use the same letter for the particular as for the general but to give the letter a subscript. For example x_0 might stand for a particular value of the variable x . This notation marks both the difference and the similarity between the variable and the unknown particular. The teachers in one of the groups with which I met, however, said that they avoided using subscripts because students found them confusing.

The other group of teachers achieved more immediate success with the task. Some had used the same notation as the students and some had chosen another letter to stand for the x coordinate of the point of contact. (In particular, David had used a in this role, whereas Katherine had used it in the generalisation of the task to the curve $y = e^{ax}$. In discussion afterwards it became apparent to David that they were using a in two different ways and he asked Katherine 'What did you use a for?'. She replied by vindicating her choice of a rather than any other letter in this context. I enjoyed this example of the ambiguity of the English language, especially in the context of our discussion of the ambiguity of mathematical language.)

In the next session I asked the teachers to comment on why this problem was hard.

Katherine Yes, it's the fact that it's a tangent going through a point that's not on the curve that they find difficult isn't it? If you ask them to find the equation of a tangent at a point on the curve they'd have no trouble

Valerie because they see that frequently, do you think?

Irene but also because they've got a point of reference that they can actually work with

David: It's awkward to visualise - it's not the easiest thing to think 'that's what it would look like, that's what the curve would look like'

John: You need a different symbolism these x s

Liz: What do you mean by that John?

John: If you use x and y , small x and small y as a sort of standard thing to use you actually get yourself into quite some difficulties and you've got to think, you've got to invent some different symbols, capital X and capital Y or something or h and k or whatever so in a sense you've got to use the symbolism, be aware of the symbols and what they mean

Two important points arise for me from this discussion. The first is the recognition that one reason for the difficulties experienced with this problem is the lack of 'a point of reference'. Students are used to finding the equation of a tangent to a curve at a particular point ('because they see that frequently, do you think?') but here they have to deal with the idea of a tangent at an unknown point.

The second is John's concern with notation. He is stressing the difficulties which arise through using x (and y) ubiquitously, so that students are not aware of 'the symbols and what they mean'. This point was raised more fully in 'The teacher's dilemma', chapter six.

The teachers' work on this task in both groups was characterised by animation and discussion. Both groups returned to a discussion of it at later meetings and two individuals mentioned it as, for them, the most significant part of our series of meetings because it marked the point at which they felt they understood what I meant by 'roles of variables' and why it was an important issue. I account for this by remarking that the roles of variables within this problem are sufficiently complex for the teachers to experience for themselves some doubt and confusion in their solution process.

Students working on Tangential II

A number of students worked on a similar task at my request. I adapted the task because some of the students had not yet come across the exponential function $y = e^x$. The adapted task was

Problem R Find the equation of the tangent to the curve $y = x^2 + 1$ which passes through the origin

Paul

One of these, Paul, expressed the same feeling as some of the teachers about being asked to find the equation of a tangent at an unknown point. He also went a bit further in exploring possible ways of getting started on the question (the transcript extract is from 'Paul' lines 105 to 116).

Liz: Hmm. Is this question harder than it looked?

Paul: Hmm. Yes.

Liz: Can you put your finger on why it's difficult?

Paul: Because you've, you've got no place to start after you've done that. You know that it passes through the origin, it could do that, it could do that (*he indicates lines passing through the origin with different gradients*) whatever, be a tangent to the curve and pass through it. I don't know, you can only do it one. Well you know it passes through the origin but you've got no, no idea where it touches and you need to know where it touches to be able to get the gradient and you need to have the gradient to know where it touches. So you've got a loop which you can't, you can't solve that easily.

Liz: Right

Paul: No it's hard

From my standpoint I can understand Paul's description as saying that this question requires an analytic rather than synthetic approach. An 'analytic' approach works from the unknown, treating it as known in order to discover its value (see my reference to Lins 1992 in chapter two). A synthetic approach starts from the known and works towards the unknown. Paul wanted to work from the known to the unknown but found that he could not do so. I tried to encourage him to use an analytic approach by saying (transcript lines 117 to 120)

Liz: Let me give you a general strategy. Working on a problem like that, I can't get anywhere without knowing something, a piece of information and I've got no way of finding it out. One strategy which will work in most cases, quite often it does work, umm is to pretend that you do know it

Paul: and then work and see if it works.

I intended him to name the unknown and produce equations which he would be able to solve to find a value for the unknown. The naming of the unknown is an expression of awareness of its importance in the structure of the problem. The production of equations is an expression of generality, of an awareness of the structure of the problem. Paul had already expressed the need to know either the gradient or the coordinates of the point of intersection so I felt that he was well placed to acknowledge the importance of these quantities by labelling one of them as an unknown.

However he understood my suggestion as advising him to guess a particular value rather than to express ignorance by the use of a letter. He eventually solved the problem by a guess and check procedure, beginning by guessing a value for the gradient. Next he found the x coordinate of the point on the curve where the gradient was equal to his chosen value. Then he checked that this point on the curve also lay on the line through the origin with chosen gradient. (transcript lines 127 to 145)

Liz: Could you describe the checking process that you're going through.

Paul: Oh yes, sure.

Liz: I think I know what you're doing.

Paul: What I'm doing is that I'm taking erh I'm giving the line a gradient, like say I'm going to do it with 2, then I'm using, I'm differentiating that to find where the gradient would equal to it - which would be the x -point - 1 in this case, then I'm putting the 1 into that (*indicating the curve $y = x^2 + 1$*), so y would equal 2. Then I'm going back to my equation on the line with the gradient 2, passing through the origin and seeing if that value I get for y there is the same as that value I get for y and putting it through the equation of that , to see if they match. 1 times 2 .. $y = 2x$.. it does, it does match that. .. $y = 2x$ so y would be 2, x was 1, $x^2 + 1$, $y = 2$ Yes that's right.

Liz: Right. So that point's on both these lines.

Paul: Yes, and it passes through the origin

Liz: and the gradients match.

Paul: Yes so the gradients to the equation of, if I answer to the question, the equation is going to be $y = 2x$, - no x is going to, we've got a gradient which is - $y = 2x + 0$. Yes, I think that's right.

In using this guess-and-check procedure he is following an approach parallel to that labelled the 'One Way Principle' by Meissner (Meissner 1979) and discussed by me in chapter two. The 'One Way Principle' advises working in the intuitive direction on a problem (in this case from the gradient of the tangent to the coordinates of the point of tangency to the equation of the tangent) rather than using inverse operations.

Paul has been through this guess and check procedure twice, first starting with a gradient of 1, which he found did not fit all the conditions, and secondly with a gradient of 2, which did. His first step in each case was to find the x coordinate

of the point on the curve which had the given gradient. In other words, having decided on a trial value for m he used it to find a value of x which fitted certain conditions. A little later in the conversation I encouraged him to develop this approach into an analytic one, where the unknown gradient is named as m and the testing procedure is adapted to set up equations from which the value of m can be calculated (transcript lines 162 to 173).

- Liz: Well, try this instead. You've picked a gradient, first you tried 1, then you tried 2. Instead of doing that, let's try m .
- Paul: m ?
- Liz: Hmm, for the gradient.
- Paul: Okay.
- Liz: Erh and work through the same thing as you did but using m , and in the process we should be able to find out something about m .
- Paul: So write an equation where I can put m in.
- Liz: Well .. umm ... yes, think back to what you did to start with. You said 'suppose the gradient's 1,'
- Paul: oh, yes.
- Liz: Now go back to that stage and think 'suppose the gradient's m .'
- Paul: If you put $y = mx + c$ (*inaudible*). You know it's zero, you know it's $y = mx$
- Liz: yes
- Paul: equals, ah $m = 2x$ doesn't it, because the gradient's - m is going to equal $2x$, so it's $y = 2x^2$

His earlier reasoning was along the following lines (although I have no evidence that his mental image was in terms of equations and implications)

$$\begin{array}{llll} \text{gradient} = 2 & \Rightarrow & 2x = 2 & \text{at point of contact} \\ & & x = 1 & \end{array}$$

With m as gradient however he proceeded in this way

$$\begin{array}{llll} \text{gradient} = m & \Rightarrow & 2x = m \\ & & m = 2x \\ & & \text{tangent is } y = mx & \Rightarrow y = 2x^2 \end{array}$$

If Paul had followed his guess and check procedure, his first stage would have been to say that the point on the curve at which the gradient is m is given by $2x = m$, so that $x = \frac{m}{2}$ at this point.

However, rather than finding x in terms of m , that is treating m as the known and x as the unknown, he took an expression for m in terms of x as his next stage. This led him into difficulties because the x he was working with here was the x coordinate of the point of contact rather than the x coordinate of any point on the tangent. Seeing that he had derived $y = 2x^2$ as the equation of the tangent alerted him to the need to rethink.

I can account for this change of direction (from x coordinate of point of contact in terms of gradient, to gradient in terms of x coordinate of point of contact) in several ways.

In his trial and error stage Paul was working from known to unknown. The gradient had been given a particular numerical value but the x coordinate of the point of contact was as yet unknown. The task, then was to express an unknown quantity, named by a letter, in terms of a known quantity. In the second case the roles of gradient and x coordinate were by no means so clear. Both were denoted by letters rather than numbers. There was nothing in the notation to indicate which was known and which unknown (in fact both were unknown, though m was to be 'taken as known' for the moment).

The task was to find the value of the x coordinate of the point of contact. If he had proceeded in a way similar to his guess and check procedure, Paul would have arrived at $x = \frac{m}{2}$. He would have expressed x in terms of m rather than finding a numerical value for it. It might have been difficult for Paul to recognise this achievement as being equivalent to 'finding x '.

In this problem x is the familiar variable and m the less familiar. Finding m in terms of x can feel like moving from unknown to known, whereas finding x in terms of m does not feel like finding anything.

In short, Paul has not recognised the role of m as 'pseudo-known' (that is a quantity which is unknown and in the long term to-be-found but which, for the present I act as though I know). The structure of the solution method which I am recommending to him is to express other quantities involved in the situation in terms of m so that I can form an equation or equations in m by considering the relationship between these other quantities. I can then solve this equation in order to find a numerical value for m . It is not clear that Paul

has the same structure in mind. At the least his grasp on this method is not sufficiently robust for him to work through the first stage.

Having taken a different turn in his work with m as gradient, Paul was unhappy with the equation he derived (transcript lines 173 to 191):

Paul: equals, ah $m = 2x$ doesn't it, because the gradient's - m is going to equal $2x$, so it's $y = 2x^2$

Liz: .. hmm

Paul: because that's $2x$ times x , is that going to work?

Liz: What do you think?

Paul: I, I don't see why it shouldn't do, because you've, you've got oh no it's going to be $2x.x$ isn't it? - as opposed to $2x^2$ because you're going to want the $2x$ times the x . .. But that's what your gradient is. It is, yes. and if we could put that into the form, if we put that into having a y , could we do simultaneous equations to try and find out where they're the same, so you had That, is that $2x$ times x different from $2x^2$? It is isn't it?

Liz: Well yes and no. Umm

Paul: As it's minus - no it isn't

Liz: What's giving you the suspicion that they're different sorts of x s?

Paul: .. It's because one has been differentiated, one hasn't - so you have an x^2 there, maybe it's been distorted somehow or other.

Paul reached the equivalent point in the working ($y = 2x.x$) that Lesley got to in hers ($y = \frac{1}{x}x$) and had a similar feeling of unease but without having the knowledge and experience to identify where the unease stemmed from.

When I asked him directly about the difference between the two x s he expressed it as 'It's because one has been differentiated, one hasn't'. Paul's main experience to date of differentiation had been in the context of finding the equations of tangents. In this context the gradient function is usually evaluated at a particular point as soon as it is obtained. The expression $2x$ may have had the status of 'particular value' rather than 'variable' in Paul's mind. He may have associated this distinction with the process of differentiation because, in his experience, the particular value had arisen as a result of a process involving differentiation. This connection between evaluation and differentiation may have contributed to his suspicion that differentiation was the key difference.

Lesley's solution to her unease was to replace x in $\frac{1}{x}$ by a different letter. This action is equivalent in some sense to naming the x coordinate of the point of contact as an unknown particular value. Paul had already chosen, under my guidance, to name the gradient of the tangent as an unknown, so this route was not obviously open to him and instead he returned to using $y = mx$ as the equation of the tangent.

Neil

Another student, Neil, also worked on this problem with me. He drew a diagram showing the two tangents which fulfilled the conditions and the conversation continued as follows (the transcript extract is from 'Neil' lines 104 to 106):

Neil: .. I mean the obvious step is to differentiate if, if I knew some relationship about what sort of gradient that (*indicating the tangent*) had, I could differentiate and say it equals that .. and the question is, at what, at what point does it cross or .. because we know this point (*the point where the curve crosses the y axis*), that's 1. .. Because of, I don't know, we need an idea of where they cross. (*he writes $\frac{dy}{dx} = 2x, y = 2x$*) I know it's going to be, Oh that's it, yes, because it doesn't go, it goes through the origin so that's just - it's that because it goes through (0, 0).

Like Paul, Neil found that information he wanted in order to get started was not given ('if I knew some relationship about what sort of gradient that had', 'we need an idea of where they cross'). He began then by taking the 'obvious step', that is by differentiating the equation of the curve. Having obtained $2x$, he treated this as a function giving rise to a straight line rather than as a gradient function for the curve.

Sam

Another student, Sam, made a similar start (the transcript extract is from 'Sam' lines 125 to 127).

Sam: Just shifted up one (*he is speaking of the parabola $y = x^2 + 1$ as compared with $y = x^2$*) and two possibilities (*he has drawn a diagram showing the curve and now adds two tangents through the origin*) there (*he writes $\frac{dy}{dx}(x^2+1) = 2x, y = 2x, y = -2x$*) Umm is that right? No it's not.

Liz: Where did it come from?

Sam: Well to get sorry umm it's not the gradient function is it? No (he writes $y - y_1 = m(x - x_1)$)

Sam also began by making a slide from the gradient function ($\frac{dy}{dx} = 2x$) to the equation of the tangent ($y = 2x$). I have identified a number of factors that may contribute to this slide in meaning.

One factor is that the gradient function which had been obtained was a linear function. If I have begun by differentiating because the word tangent has suggested this course of action, but without any very clear idea of what will follow, then I might seize on $2x$ as my answer because it is of the right form.

A second factor is that the slide from gradient function to tangent function can be achieved by missing out two stages in the familiar procedure for making this move. This familiar procedure is as follows:

- (i) differentiate the equation of the curve to get the gradient function
- (ii) substitute the x coordinate of the point of contact into the gradient function to find the gradient of the tangent
- (iii) substitute this gradient for m in the form $y = mx + c$
- (iv) find the y -intercept for the tangent

Neil and Sam's slide omitted stages (ii) and (iii).

A third factor is that the equation of the tangent which is sought must be of the form $y = mx$ since the tangent passes through the origin. Having this in mind at the moment of seeing $\frac{dy}{dx} = 2x$ makes it possible to seize on $2x$ as 'the answer' without considering the intervening steps.

The slide from $\frac{dy}{dx} = 2x$ to $y = 2x$ involves a shift in the meaning of x from 'first coordinate of any point on the curve' to 'first coordinate of any point on the tangent'. This shift appears to have been made unconsciously and unintentionally by Neil and Sam.

By an unfortunate coincidence this slide in meaning can result in the right answer to this problem! Neil and Sam both eventually reach this point again by a more satisfactory route.

Neil quickly decided that this method for finding the equation of the tangent was not rigorous and looked for another way. I suggested naming the unknown point of contact and he labelled it (a, b) . Using this as a starting point he worked out that the equation of the tangent is indeed $y = 2x$. Afterwards I asked him about the difference in the two methods (the transcript extract is from 'Neil' lines 169 to 176):

Liz: Ah hmm. Umm you had the erh equation $y = 2x$ in your second line.

Neil: Hmm, yes but,

Liz: umm, so why couldn't you have said it there?

Neil: Because that's the equation for, that tells me what if I've got a value of x , what the gradient is at the point. Not an equation for the line. I got that from differentiating the original equation and not from .. umm, not from the knowing where the points were

Neil was then very clear about what the status of his first $2x$ was. Now that he had the benefit of an overview of a solution procedure he could stand back and see what each stage was achieving. In the moment of exploration and uncertainty, when he first thought about the problem, the slide was possible.

Sam, similarly, found it possible almost immediately to be lucid about why this approach was not valid ('it's not the gradient function is it?').

Tommy

The elision between $2x$ as the gradient function and $2x$ as the straight line was also an issue for Tommy (the transcript extract is from 'Tommy' lines 20 to 70).

Tommy: so, so we know that the equation I'm trying to find should be $y = mx + c$ (*writes $y = mx + c$*) m is the gradient, and c is the y intercept - so it will be $y = mx + 0$ (*writes $y = mx + 0$*) um, so $2x$ differentiated will come out to replace the mx (*writes $2x = m$*).

Liz: Say that again. (*I want to clear up the anomaly between what he said and what he wrote*)

Tommy: You replace the mx with this $2x$. (*writes $2x = mx$*)

Liz: Right. Replace it?

Tommy: Yes.

Liz: Why are you doing that?

Tommy: Umm because that's the gradient function

Liz: What do you mean by gradient function? (*I want to find out whether his understanding of differentiation includes substituting a particular value of x to find the gradient at a point*)

Tommy: Gradient function, .. the $2x$, if you have any point on the curve - so they're like $x = 2$, then substitute into this equation $2x$, 2 times 2 is 4, that means the gradient is 4 when x is equal to 2 - on this curve

Liz: Right. Okay. So how does that tie in with this mx ?

Tommy: Well the gradient is going to be equal to the curve's gradient - at the point of intersection. So that will be $2x$ is equal to mx . (writes $2x = mx$ again)

Liz: Ah ha. Umm .. mx is the gradient then is it?

Tommy: No. (inaudible) Is that right? (he crosses out the last two equations)

Liz: No.

Tommy: Well,

Liz: But do you know what's wrong with it? What's the gradient of this line?

Tommy: This line? (pointing to the tangent)

Liz: Hmm.

Tommy: m .

Liz: Hmm. Not mx ?

Tommy: No. $2x$ is equal to m .

Liz: Ah hmm. Where is that equation true?

Tommy: At the point of intersection.

Liz: Right. Okay.

Tommy: So I can rearrange that to find so $y = 2x + 0$. (writes $y = 2x + 0$)

Liz: What's that?

Tommy: This?

Liz: Yes, what's that, the equation? Where did you get that from you've written down?

Tommy: In the m is equal to $2x$. Then substitute that into the general equation. $y = mx$ becomes $y = 2x$.

Liz: y is equal to mx (stressing the x).

Tommy: Oh yes. that means um substitute that in to $y = .. 2x$ times x (writes $y = (2x)x$) That can't be right.

At many points in this conversation I was confused as to what Tommy's understanding was. As I questioned him he seemed to become less confident and more focused on saying what I wanted to hear. Although he was able to rehearse the meaning of m in the equation $y = mx + c$, (m is the gradient, and c

is the y intercept') he did not distinguish between replacing m by $2x$ and replacing mx by $2x$ until I forced him to do so. He acted as though he understood mx as the gradient, yet when I asked him the gradient of the line he replied that it was m . Like Paul, he realised that something was wrong when he was faced with $y = (2x)x$ as the equation of the tangent.

As I questioned him I found that his understanding of the constituents of this problem was secure. He gave clear answers concerning the meaning of $y = mx + c$ and the gradient function. My interpretation of his work is that the reason for his slowness in solving the problem (he did reach a solution eventually) is his lack of experience in addressing the roles played by the variables in the problem. The central awareness required to solve this problem is that the point of contact is unknown and must be found. Tommy, however, did not acknowledge the particular-unknown status of the coordinate of the point of contact either in his treatment of it or in his notation. He neither set about forming equations which he might solve to find this coordinate, nor identified it as of interest by labelling it with a notation other than x .

One of my intentions in this chapter has been to show that this awareness of roles of variables is a vital ingredient in the ability of students to tackle second variable problems.

In deriving the equation $y = mx - mx_1 + y_1$ an awareness of the role of each of the variables contributes to successful derivation and use of the final form. The roles of m and c as placeholders, to be replaced by expressions in the final form were discussed in story one; the roles of x_1 and y_1 as particular-to-be-taken-as-known, to be worked with as though they were particular numerical values were discussed in stories one and two; the roles of x and y in representing first the coordinates of every point on the line, then, by substitution, a particular point on the line, and finally again any point on the line, and as having an essential role in the equation of a straight line were discussed in stories two, three and four.

In the tangential problem an awareness of the difference between the general point on the curve or tangent and the particular point at which they meet was seen to be a crucial aspect of understanding the problem.

I have pointed in this chapter to other factors which might influence students' actions if the awareness of these roles is not sufficiently strong. I have also suggested some strategies which teachers might use to increase students' awareness or reduce the influence of the other factors.

These suggestions formed part of my first purpose in this chapter, which was to indicate how an awareness of the issues I have raised in chapters five to seven might inform classroom practice.

Chapter 9 Summary and Conclusions

In this chapter I present by way of a summary the main themes which I have developed through chapters five to eight. I then refer back to the questions I raised in chapter two and go on to consider directions for further research. Finally I look back to the question of validity in the light of chapters five to eight.

Summary of Themes

Particular and General

In chapter five I presented two models of generalisation as I understood them from the writings of others. One model was of abstraction from many particulars, whilst the other involved a direct grasp of the general as a result of experience of only one particular. I also distinguished between empirical and structural generalisation by looking, in a number of problems, at different means of arriving at a generalisation. I described four types of generalisation as follows:

- An 'inductive generalisation' is the result of inductive rather than deductive reasoning, that is it generalises beyond the domain in which the proposition is already established
- An 'empirical' generalisation is one made primarily on the basis of results as opposed to process
- An 'on-the-spot' generalisation is one that is made on the basis of only one example

- A 'structural' generalisation generalises a result from a single or several examples based on the generalisability of the process by which that result was obtained

I examined a number of situations in which students, colleagues, or I myself had made generalisations and tried to understand them in terms of the distinction between generalisations made from one or from many examples, and in terms of the distinction between empirical and structural generalisation. I found that the empirical/structural distinction was more relevant to my experiences of generalisation, that is, in terms of informing my teaching actions, it was more useful to focus on the use that a student was making of examples than on the number of examples that were used.

When used by experienced mathematicians, in the work reported in chapter five, inductive generalisation was accompanied by other grounds for conviction, whereas students were less able to draw on other confirming arguments. I found that for a few students, actions designed to lead to inductive generalisations (i.e. producing a table of 'results') had become the predominant approach to problem-solving and were used in inappropriate settings. One conclusion from the discussion in chapter five is that the generic example is a possible means of enculturation into the practice of structural generalisation.

Again in chapter five I proposed that the expression of generality is not frequently required of students doing a course in 'A' level mathematics. Students who are 'given' expressions of generality rather than deriving them may not understand them as expressions of generality, that is they may not see the particular in the general. The form then has no content, that is the symbols which for the expert are representatives of a set of possible particulars, for the student represent only themselves. Further I claimed that the involvement of more than one variable in expressions of generality makes a significant difference to the level of difficulty of the task. My experience of using the classroom gambit 'Particular, Peculiar, General' suggests that students can improve their ability to express general statements with practice.

The task of placing a general point on a diagram raised the issues of agent of choice and of freedom and constraint. In the context of locus problems there is an ambiguity connected with placing a point which is to be understood as general (i.e. as (x, y) or as (a, b)) on a diagram. Such placement may be experienced either as allocating particular values to the coordinates of the point or as expressing a relationship between them. The difference between these two intentions is not visible to an observer and is not adequately distinguished by

the language, 'it could go anywhere', frequently used in these situations. The question of agency of choice was raised again in chapter six where I was concerned with choices made about which letter to use to stand for certain variables. Here I distinguished between mathematically necessary choices and culturally determined choices. Mathematically necessary choices follow from the setting up of the problem situation whereas culturally determined choices are dictated by the learner's previous experience of such situations and the roles which they have come to associate with certain letters. My conclusion was that students may experience culturally determined choices as mathematically necessary. Teachers' awareness of this possibility will shape the way in which they approach the conventional use of letters.

Roles of Variables

In chapter six I proposed that the roles of variables within a problem situation can be viewed in terms of their order of variation and that different orders of variation represent different ways of seeing the situation. Consideration of the possible orderings of the variation can give the 'expert' a way of generating different ways of seeing and of anticipating the students' possible views. I rejected the notion that each problem has one correct order of variation in favour of the contention that alternatives to a possible conventional ordering induce different perceptions and can give alternative methods of solution.

Later in chapter six I identified, from my own experience of learning mathematics, situations where a shift in the meaning of certain letters took place. This shift was from being understood as representing a particular although unspecified (or unknown-to-be-taken-as-given) quantity to being understood as representing a variable. The situations in which this shift became apparent for me were those where, in the final stage, the variation of the quantity could be seen to generate a locus. Here the letter was first treated as representing the coordinate of (or value of the parameter at) a particular point on the curve and secondly seen as representing the value at any point on the curve. In my working on these for the first time as a student, the shift was marked by confusion, unexpectedness and finally pleasure as my understanding progressed. The pleasure was a result of the experience of versatility and control in my ability to alter my perception of the role of the letter. With practice my conscious experience of the shift vanished and it became difficult for me as expert to re-enter the learner's experience. My aim in presenting tasks and accounts has been to allow others to recognise the novice's confusion. The emotion and energy attached to this learning experience for me, which allowed me to recall it from a distance of nearly twenty years, suggest that it is a shift of some importance.

In chapters six and seven I illustrated and described two aspects of the tension between fluency and awareness. The first aspect is that of choice of literal symbols. Cultural conventions in the use of certain letters in certain roles remove attention from the choice. This can assist fluency and free attention for other tasks. On the other hand it can hide the possibility of choosing differently and thus mask the arbitrariness of the choice.

The second aspect is that of the action sequence solve-and-substitute-back. I observed this sequence in operation in the work of a number of students. For some the frequency of the student's encounters with this sequence developed fluency which was of benefit in terms of speed and freed attention. However on occasions the automaticity developed resulted in the sequence being used in unhelpful ways. The substitution was made into an inappropriate equation or the sequence was used to obtain an expression for a variable which was not of central concern. Fluency militated against conscious decisions as to these points.

In each case the teacher's dilemma is to balance the benefits of fluency against the need for awareness. The teacher's dilemma might be understood as a question of how much practice of the conventional situation is needed before it is possible to experience a situation as different from the conventional. The teachers' discussion focused on the stage at which different letters could be introduced into familiar situations. However this understanding assumes that practice is the best or only route to fluency and it is not the only way of framing the question of the balance. One alternative view is that fluency is best achieved through subordination and removal of attention (see, for example, Hewitt 1994).

In the final section of chapter six I explored the role of the unknown-to-be-taken-as-given, a variable which must be worked with as though it were known. Many situations which arise in studying 'A' level mathematics require the student to express an unknown in terms of these variables. I contrasted the experience of the expert mathematician, who can describe arriving at such expressions as 'finding' the variable, with those of the novice, who does not experience any increase in knowledge or certainty through achieving such an expression. Language patterns used by two students in these situations express more consciousness of not knowing than of knowing. Their experience of expressions for an unknown in terms of an unknown-to-be-taken-as-given can be compared with that of a student who experiences 'lack of closure' in the expression $2a + b$. For the teacher, an anticipation this response from students allows the preparation of appropriate teaching actions.

At the end of chapter six I summarised four kinds of problem in which the roles of the literal symbols change as the solution proceeds. The four types of shift were as follows:

- from variable to unknown-to-be-found, for example, in a solution to the question 'Find the coordinates of the point where the line $x + 2y - 4 = 0$ meets the line $y = 2x - 2a + b$ '.
- from the role of placeholder within a form to that of unknown-to-be-found, for example in answering the question 'What is the equation of a straight line with gradient 3 which passes through the point (2, 8)?'
- from unknown-to-be-taken-as-given to unknown-to-be-found, for example, in solving analytically the problem 'Find the point of contact of the tangent to the curve $y = x^2 + 1$ which passes through the origin'
- from unknown-to-be-taken-as-given to variable, for example in solutions to locus problems such as, 'A point P, coordinates (a, b) is equidistant from the x -axis and the point (3, 2). Find a relationship connecting a and b .'

None of these four shifts is apparent in the problem solution which the solver writes on the page. They are not expressed by the ordinary rules of algebraic manipulation but constitute reasons for writing down equations. As such they demand awareness, not technique. Each represents the rationale behind an action sequence which is an important element of mathematical fluency at 'A' level.

Labels

In chapter seven I discussed a group of incidents which had led me to the label 'losing track'. I compared my experience of bewilderment in the middle of working on a problem with what I had observed in the work of two students. The recognition and labelling of such an experience allowed me to contemplate future actions in similar circumstances. It also allowed me to consider the awarenesses which I had called upon to release me from the state of having 'lost track'.

The label 'losing track' relies on a metaphor of a journey applied to the process of problem-solving. This metaphor also underpins a further distinction, which I made in chapter seven, between planning a problem solution and responding to triggers as the solution progresses. These triggers might include the 'forms' which are taught in 'A' level mathematics courses, for example $y - y_0 = m(x - x_0)$. In particular the superficial visual form of an equation may trigger an action. By contrast, an anticipation of the form of the final answer encourages the planning of a solution procedure which will arrive at that form.

At a different level of meta-cognition I can use an awareness of the label 'of the form' to provide alternative ways of seeing my way through a problem.

In chapter eight I raised many of these issues again in the context of two particular problems. In part this chapter is a demonstration of how the themes of chapters five to seven can inform a reading of students' responses to tasks. In addition I made some suggestions, in the first part of the chapter, for teacher actions which might address the awarenesses which I had identified as important. Each of these actions amounts to an attempt by the teacher to enable the students to shift their attention, that is they are invitations by the teacher to see things differently. They are ways in which the teacher can help students to move from novice to expert awareness of the mathematical situations in which they are working.

In the second part of the chapter I gave an example of a task which proved a very rich one for me and for the two groups of teachers who worked on it. It is included as backing for my assertion that working on mathematical tasks can be useful in enabling teachers to think freshly about their own classroom practice. It also serves as another problem setting within which to examine, through what they say as they are working, students' awareness of shifts in roles.

'Second Variable Thinking'

In chapter two I raised the question of whether it would be useful to arrive at a meaning of 'parametric thinking' along the same kind of lines as the meanings which have been suggested for 'algebraic thinking'.

I am now in a position to set out some of the factors which are involved in an understanding of mathematical situations which involve second variables and I prefer to describe such understanding as 'second variable thinking' rather than 'parametric thinking' for the reasons I outlined in chapter two. Six of the themes which I have drawn out of my experience as related in chapters five to eight contribute to such a description. I offer these as the beginnings of a list of mathematical awarenesses connected with second variable situations.

- First on my list is the ability to see the particular in the general, that is to interpret an expression of generality in a particular case. An example is a recognition of $x^3 - 8$ as being of the form $x^3 - a^3$.
- Second is awareness of the need for more than one letter in some expressions of generality. In many cases such an awareness includes an understanding of the different roles played by these different letters (whereas in others the roles of the different letters are in fact the same). For

example in the expression $p(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for the Binomial probability function, an understanding of the need for both n and x is such an awareness.

- Third is an awareness of the role of a 'general point' in establishing the equation of a curve. This includes the ability to distinguish between attributes of the point which are due to its general nature and those due to its particular position, and may be evidenced in a willingness to 'pretend' that a particular point can be perceived as general.
- Fourth is an ability to exercise control over the choice of letters to be used in a problem situation. I have described the difference between mathematical necessity and cultural determination as motives for these choices and suggested that successful dealing with second variable situations requires an ability to distinguish between these two.
- Fifth is an ability to treat a literal symbol as an unknown-to-be-taken-as-given, that is to treat it as though it were a number. This means valuing the expression of one letter in terms of another as being potentially useful, rather than containing no information. By contrast, students frequently misinterpret literal symbols as having the roles of variables (being able to take all real values and having importance only in their relationship with another variable) so that, for example, they do not distinguish between x and y and p and q in solving the question 'Find the equation of the straight line with gradient m and passing through the point (p, q) '.
- Sixth is an ability to make shifts in the interpretation of literal symbols during the course of a problem solution. The four kinds of shifts which I encountered in this work are outlined above. Each represents an important capability in dealing with variables.

This list of six awarenesses is offered as a first step in the process of establishing what might be meant by 'second variable thinking'. For me the usefulness of such a list lies in the teacher's ability to recognise the behaviours of her own pupils in the descriptions and to use this recognition to prepare alternative teacher actions. It does not lie in the completeness or precision of the list.

Historical Parallels

In chapter two I posed some questions about parallels between the psychological and the historical development of algebraic concepts. My concern was whether it was appropriate to equate my students' first encounters with second variable problems with the move from 'algebra of a fixed value' to 'functional algebra'

(Sfard and Linchevski) or from the Diophantine to the Vietan stage (Harper). My conclusion is that there is a significant overlap between the types of problem identified by these authors as representing a move from one stage to the other and the second variable situations I have studied, but that they do not completely coincide. For example, the question 'Which is larger $2n$ or $n + 2$?', which is used by Kieran (1994) to illustrate the 'functional algebra' stage, is not a 'second variable situation' in my terms.

The 'historical stages' framework gives me a way of seeing my students' algebraic development and a way of explaining, in general terms, the difficulty experienced in moving from one stage to another. However I feel that the insights I have developed into the nature of the awarenesses required to deal successfully with second variables are more productive in terms of suggesting alternative teacher actions. I have demonstrated in the first part of chapter eight how the 'stories' I have used to describe students' behaviour can generate possible teaching strategies.

Directions for Further Research

Each of the themes which I have developed in the thesis and summarised in the first part of this chapter could be expanded upon or looked at in a different context. I have chosen to refer to just three of these themes and describe briefly what this might entail.

First, there is scope for further exploration of the idea of a generic example. A relatively unresearched area is that of students' perceptions of and responses to examples which are presented by the teacher with the intention that they should be perceived as generic. An extension of this idea to younger pupils would be a useful complement to Harel and Tall's work on principles for generic abstraction (Harel and Tall 1991).

Secondly, I consider my distinction between empirical and structural acts of generalisation to be a useful one which could be pursued in other settings. The work of younger or older students and the expectations of teachers could be examined in the light of this distinction. This would serve the purpose of testing the usefulness of the distinction beyond the immediate context of sixth form algebra.

Thirdly, it would be possible to add to and refine my list of the components of 'second variable' thinking by testing it against further examples of students' work in second variable situations. This would entail judging the capacity of the list to explain (to my satisfaction and that of colleagues) students' behaviour and adjusting it accordingly. This explanation would be in terms of awarenesses the students demonstrate or seem to lack.

Validity

My conviction of the validity of my work rests on two things. The first is its potential, which I have already experienced, to inform my own practice. I can trace some of my moment-by-moment classroom decisions back to elements of this thesis. For example, when I was about to teach a class about the cumulative distribution function for a continuous random variable I could foresee that

$$F(x) = \int_{-\infty}^x f(x) dx$$

might cause conceptual difficulties for students. I decided to organise a discussion of the nature of definite integrals before introducing it.

On another occasion (recounted in chapter six) $ax + by = c$ was rejected by students as an example of a general linear equation in favour of $ax + b = c$. I was able to take the opportunity of this unexpected response to draw students' attention to their stereotypical use of letters.

I can account for elements of my planning decisions also by looking at the work on 'forms' in chapter seven. My thinking here on the extent to which standard forms are an expression of generality for students, and on the way in which these forms can influence approaches to problem solving, shaped the way in which I worked on probability distributions with a group of students. The 'forms' in question here were, for example, the expression for the Binomial probability function $p(x) = \binom{n}{x} p^x (1-p)^{n-x}$. I introduced this expression by giving examples from which I asked students to draw out the common pattern. The way in which I did this was influenced by my desire to allow the students to see it as a summary of particulars as well as a standard form to be used in problem solving.

My second basis for conviction is the response of numerous people with whom I have shared my ideas during the preparation and writing of this thesis, especially the two groups of teachers. Their various expressions of interest, readiness to share examples from their own experience and willingness to experiment in their classrooms with what I had suggested have persuaded me that the themes and issues I was drawing to their attention were of an importance beyond my own interest and that the accounts and tasks which I devised were effective in raising awareness of these themes.

In the final analysis, however, the validity of this work is a matter for, you, the reader. The accounts and transcripts are valid if they allow you to compare with your own experience. The validity of the tasks rests in their capacity to

raise your awareness of a mathematical issue. The arguments and interpretations need to be convincing. The whole must have the potential to inform your future actions as a teacher of mathematics.

Appendices

Appendix A Developments in the use of the word 'parameter'

Use of the word parameter has developed from a narrow technical meaning in the field of mathematics to a much less specific meaning in a far greater number of fields. The Oxford English Dictionary gives as its earliest citation:

Mydorge 'Prodromi Catoptricum et Dioptricum' 3 Def xix (1631)

'Parametrum conic sectionis dicimus, rectam lineam cuiuslibet conic sectionis, aut portionis, vertice eductam ordinatim ad contiguam diametrum applicatis aequidistantem ... Quae, si ab axis termino sit educta, recta parameter .. dicetur.'

'We call the parameter of a conic section a straight line from any conic section, or part of it, produced in order from the vertex touching the diameter equidistant to those lying near Which, say, if it were to have been produced from the end of the axis will be called the proper parameter.'

(I am grateful to Graham Jagger for this translation)

The OED's more succinct definition is 'In conic sections: The third proportional to any given diameter and its conjugate'

An explanation of some of the technical terms used in, or required as background to, this definition might be useful:

A *chord* is a line segment joining any two points of a curve.

For any conic section there is one point which bisects any chord drawn through it. This point is called the *centre*. (The centre of a parabola is at an infinite distance)

The midpoints of a set of parallel chords lie on another chord, known as a *diameter*. Every diameter passes through the centre and every chord passing through the centre is a diameter. (The diameters of a parabola are all parallel to each other and to the axis of the parabola)

The parallel chords bisected by a diameter are known as its *ordinates*.

Of the set of ordinates of a diameter, one passes through the centre, i.e. it is also a diameter. Diameters which are each others ordinates are called *conjugate* diameters.

The *parameter of a diameter* is the third proportional of the diameter and its conjugate. (p is third proportional to a and b if $p = \frac{2b^2}{a}$)

The *parameter of a curve* is the parameter of its transverse axis.

The parameter of a conic section is also equal to its *latus rectum*, which is the length of the chord parallel to the axis which passes through the focus.

This technical use of 'parameter' is now obsolete and the OED cites no uses of it from the twentieth century.

A quotation from the nineteenth century shows how this technical meaning may have begun to develop into a more general use as 'a quantity which is constant (as distinct from the ordinary variables) in a particular case considered, but which varies in different cases'.

Lacroix Differential and Integral Calculus 1816

'If ... the parameter of a parabola be made to vary, a series of parabolas will be obtained.'

The role of 'parameter' as the controller of or key to variation within a set or family of parabolas is extended to other curves in a later quotation.

Price Infinitesimal Calculus 1852

'If an equation to a curve be given, involving one or more constants, as well as the current coordinates, the position and dimensions of the curve will be changed by a change in the constants, and yet the class may remain the same ... A constant that enters into an equation, and varies in the way above explained, is called a variable parameter.'

The word parameter also developed a somewhat separate meaning as a descriptor of variation *within* a given curve:

Grace and Rosenberg Coordinate Geometry xvi. 220 1907

'If we can find simple expressions for the coordinates of points on a conic in terms of one variable quantity, a point on the curve may be looked on as determined by a definite value of the variable, the variable being usually called the parameter.'

In the nineteenth century the word took on meanings within the fields of optics, astronomy and crystallography and later in computing, statistics and electricity. By the late twentieth century we find 'parameter' used very widely in discussion of politics and current affairs to mean aspects, limits or characteristics of any situation:

'Time' magazine 3 August 9th 1970

'The fact that Nixon was willing to make his chastisement public suggests ... that the President at least understands the parameters of the problem.'

'Today' programme Radio 4 August 1996. Interviewee discussing the postal workers' dispute

'All the parameters are on the table ... '

Appendix B Early experiences of the two schools

School B

I wrote formally to school B's headteacher and head of mathematics (HoM) saying very briefly what I hoped to do and that I would phone the next week to see if they could accommodate me. When I spoke to the HoM the next week he said that he would not have considered responding to my request if he had not had a positive report about me from a colleague. I asked if I could come in and observe a sixth form lesson being taught and discuss possibilities and we arranged a time. During that visit I spent an hour talking to the HoM but did not get to see any teaching. He said that he was interested in my proposal but would need to get the agreement of other members of the department who would be involved, namely the teacher whose class I would probably work with and the senior teacher in charge of mathematics in the sixth form. He suggested that I should make a further visit which I should view as an opportunity to persuade them that it would be useful for them to have me working with them. I visited the school again a few days later by arrangement and spent about half an hour talking to the three of them about my research and what it would involve them in. They were all attentive and interested and made positive comments about what I had to say. A few days later the HoM phoned me to say that he would be happy for me to begin working with them in September on a trial basis. I arranged another visit before the end of term so that I could see Nigel at work with a sixth form class. I spoke to the HoM again who said that he would ask Nigel to phone me during the summer to confirm the timetable slots during which the sixth form would be taught.

Having heard nothing from Nigel during the summer I phoned the school the day before I was expecting to start (assuming no timetable changes) but could not speak to him. I left a message to say that he should expect me the next day. I went in the next day to see what I thought would be the first lesson with that class but found that it was the second (there had been a timetable change). Nigel did not appear to have received my message. The lesson was spent in working on a worksheet which had been set as holiday work during the induction course at the end of the previous term. It dealt with factorising expressions and solving quadratic equations. The lesson was fairly formal. I was introduced as Miss Binns "who will be working with us for a few months" and as an expert mathematician who could help them with their work. I helped individuals with their work during the lesson. Pupils worked individually throughout. There were no lessons the next week because of an internal school event.

During the first lesson I spoke to one boy who was apparently having difficulty with his work. He had done some questions which were a mixture of differences of two squares and sums of quadratic and linear terms (no constant term) to factorise. He had obviously had some difficulty in distinguishing the two cases and Nigel had helped him put some of them right. I asked him if he had them all right now. He wasn't sure. I asked him if he could explain to me the two different types of question that he had been doing. He didn't reply to my question but said that he wanted to get on to the next section now.

School A

My contact with school A was made by meeting up with an old colleague who is now HoM at the school. This meeting took place early in 1993 and at that time I was at an early stage in my thinking. I therefore just made a tentative suggestion which was enthusiastically received, that I might come and work at the school at some time in the future. Subsequently I (unwittingly) met another member of the school staff at a conference in April. When I made a more well-defined approach to the HoM at a later stage he responded positively. I suggested that I might come in to speak to the department, partly to keep them informed and partly to give them the opportunity of volunteering to have me with their class if they wished to. The HoM welcomed the idea, although he felt that it was already decided that I should work with Peter (whom I had met at the conference). About ten members of the department (all 'A' level teachers) were present when I visited. I made the same presentation of my work as I had at school B. Only the HoM and Peter showed any signs of interest.

I did not have a chance to visit school A to observe any teaching before the end of term. I met with Peter for an hour during the summer to discuss what we would do. I went expecting to observe the first lesson, which was on basic algebra, but got involved in speaking to the whole class when Peter had to leave the room for a while. I was introduced as Liz and then, by myself, as being a research student from the Open University, about which I would say more later. The next lesson was on factorising quadratics. I spent most of my time with individual students. The third lesson I directed with Peter present.

At school A pupils call staff by their first names and walk freely into the staff coffee lounge at break to speak to a teacher. There are no bells to mark change of lesson. There is no school uniform.

Appendix C Leaving School B

Last week I decided not to go into school B any more. My reasons:

- I felt in need of more time on a regular basis for writing, reading and making other contacts
- in some ways the school B experience could be seen as duplicating the school A experience. The structure of the relationship was similar
- School A was proving to be the more useful experience because
 - pupils responded more favourably to being asked to do something different from normal, and to being 'researched' (having tape recorders in the lesson etc.)
 - pupils were more willing to talk about their work and especially their misunderstandings
 - the teacher was more willing to indulge in talk about teaching strategies as opposed to the progress of individual pupils
 - the teacher made use of me in the class as another teacher and as a mathematician, rather than seeing me as an outsider and observer only

I telephoned both the HoM and Nigel on the Tuesday evening before Wednesday 12/1/94 which was my last visit. I had a brief talk with Nigel and a longer one with the HoM. I hoped to speak with Nigel for longer on the following day but he was in a hurry to see a pupil after the lesson. He said he wanted to thank me for my help over the last term. I tried to make it clear that he had not failed in any way and that my leaving was not a judgement of him as a teacher, but it was a hurried conversation. On the phone the HoM asked to see me for lunch which we arranged for the next Tuesday. He has asked for my thoughts on the department, and for any way in which they can help me.

My regret in leaving is any distress I may have caused Nigel and also a feeling of having given up on a challenge. Because the class were less receptive to my ideas than the school A class it would have been good to have worked long enough with them to overcome the resistance. This would have involved a lot of work on personal relationships as well as mathematical ideas.

Appendix D Transcripts

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Eddie 23/5/94

- 1 Liz Let's try starting with that one
I hand over problem 3 'Sketch $y = x(x - a)$ '
- 2 Liz I think you'll find that reasonably straightforward
- 3 Eddie Yes, so as I go through it speak out loud, yeah?
- 4 Liz Yeah just tell (inaudible)
- 5 Eddie Well first of all you'd expand the bracket x times $x - a$ so you've got 'Sketch $y = x^2 - xa$ '
- 6 Liz Uhmhm. Why have you done that?
- 7 Eddie Cos it's easier to find the powers of x
- 8 Liz Right
- 9 Eddie Cos an x squared graph would be a curve
- 10 Liz Okay
- 11 Eddie Then I'd work out the points where x is equal to zero to give me some indication of where it would cut and obviously it's x squared minus x times a so when a equals x , or when x equals a (*as though correcting himself*), it's going to cut the axis
- 12 Liz Uhmhm.
- 13 Eddie Or when x equals 0 it would also equal 0 so I know it's going to go through the origin on this axis - it's also going to go through the point a on the x -axis. Because the coefficient of x^2 is positive it means it's going to be a sloping graph from the top left dipping downwards going through both points and then curving back upwards to the right. (inaudible)
- 14 Liz Marvellous. Um .. what .. tell me about um the role of a in that equation.
- 15 Eddie Er, well the a would be the coefficient of the x . So, er, whatever value x is .. it would be $x^2 - xa$ so if a is greater than x it would be a negative number
- 16 Liz .. Right
- 17 Eddie Or well it depends if they're negative or not, but if a is less than x then it would generally be a positive number
- 18 Liz Uhmhm. What would happen if you vary what a is then?

- 19 Eddie Er, as you vary a if a is a low number - like take x is a constant, say 1, for example,
- 20 Liz Yeah
- 21 Eddie So you'd have 1 squared, which is 1, minus 1 times a which is a ... as a rises y becomes a lower number taking x as 1 .. and as x is lower y becomes higher so there's an inverse relationship between y and a in this case.
- 22 Liz Right that's when x is 1
- 23 Eddie That's when x is equal to 1, yeah.
- 24 Liz Okay, what does .. think about what, um, what you just told me, that if x is 1 then there's an inverse relationship between a and y - one goes up and the other goes down
- 25 Eddie Yeah
- 26 Liz Um, could you interpret that in terms of what you've got on the graph?
- 27 Eddie Well, it would be about the point which is the minimum so if you differentiate $x^2 - ax$ you get $2x - x$
- 28 Liz $2x - a$
- 29 Eddie $2x - a$ sorry .. $2x - a$ So when x is equal to 1, which will be 2, then a will equal 2 for it to be a minimum point
- 30 Liz Uhmhm
- 31 Eddie So about the point where a is equal to 2, higher than that will be upward on the curve and lower than that will also be upward on the curve because they're each sides of this point here. which is the minimum point.
- 32 Liz Mmm okay. Um, if I asked you to draw the curve of $y = x^2 - 2x$
- 33 Eddie Yes
- 34 Liz What would that look like? You don't necessarily need to draw it if you can describe it without
- 35 Eddie Yes. It would also be a slope as the coefficient of x^2 is positive but it would be $x^2 - 2x$ so it would cut 0 again when x is equal to 0 but also when x is equal to 2 because 2^2 is 4 minus 2 times 2 which is also 4 would equal 0
- 36 Liz Uhmhm
- 37 Eddie So the point a in that case would be 2.

- 38 Liz Right okay. Umm .. and if we made a into 5 instead, so we've got $x^2 - 5x$ what difference does that make to the graph?
- 39 Eddie It just .. pushes the graph outwards - it makes it more of a bowl shape than a 'U' shape
- 40 Liz Right
- 41 Eddie The greater the coefficient of x , the greater the bowl the curve is. Also cutting through the origin as if it's just x in the coefficient of x when x is equal to 0 it's bound to equal 0. But as a rises the second point which is x times a equals x^2 also pushes outwards
- 42 Liz It's getting further away
- 43 Eddie Yeah
- 44 Liz Right, great, wonderful - let's try another one try that one next
- I pass him problem 4 which reads 'Find, in terms of a and b , the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$ '*
- 45 Eddie First of all I'd remember a few formulas,
- 46 Liz Okay
- 47 Eddie like the gradient of the normal line which is m and the perpendicular gradient is one over, negative one over m .
- 48 Liz Right
- 49 Eddie So first I'd work out the gradient for this which would be the coefficient of x , the term, so just rearrange it, you've got the $2y = 4 - x$, divide through by 2 you've got $y = 2 - \frac{x}{2}$ so then you just have $y = \dots\dots y = 2(1 - x)$ no $y = \frac{1}{2}(4 - x)$
- 50 Liz Okay
- 51 Eddie So I think the gradient of the line would be $-\frac{1}{2}$
- 52 Liz Yeah, that's right
- 53 Eddie Yeah, cos the coefficient of x is negative so you'd make the
- 54 Liz Yeah

- 55 Eddie coefficient of that negative. So obviously if the gradient of the normal is $-\frac{1}{2}$ the gradient of the tangent will be 2
- 56 Liz Right
- 57 Eddie So, in terms of a and b the perpendicular from point a to that line what I'd do now is I'd work out what this was in terms of a and b
- 58 Liz What do you mean by that?
- 59 Eddie Er, well this perpendicular
- 60 Liz Do you know what the foot of the perpendicular means? Do you know what that term means?
- 61 Eddie No I don't to be honest
- 62 Liz Well it means, um, if .. you've got this line (*David draws a diagram showing two lines intersecting at right angles during this and the next several speeches*)
- 63 Eddie Yes
- 64 Liz And you've got this point
- 65 Eddie Yes
- 66 Liz You draw a line from the point to the other line, um and it meets it at a right angle
- 67 Eddie Yeah
- 68 Liz Can you put the point on?
- 69 Eddie (a, b) ?
- 70 Liz The point (a, b) yeah
- 71 Eddie It should be there then (*he marks the point (a, b) at the intersection of the two lines*) where the lines cross
- 72 Liz Which was your original line?
- 73 Eddie The $x + 2y - 4$ line
- 74 Liz Yeah, is that this one? (*I point to one of the intersecting lines*)
- 75 Eddie Yeah
- 76 Liz Okay. ... Do you know that this point is definitely on that line?

- 77 Eddie .. Well a and b could be any . variable, can't they, so it must be one point on the line
- 78 Liz ... well .. no, not necessarily, because as you say a and b could be anything
- 79 Eddie Yeah, they can, but let's presume they can be any number, which could fall into that category. I mean if they were five million and a half they wouldn't fit into that equation obviously but as they are variables you can just presume they will .. well, that's how I would anyway!
- 80 Liz Mmm Yeah, well in fact the meaning of this term 'the foot of the perpendicular' rather assumes that this point isn't on that line. Let me just show you what the diagram is like. It's a bit difficult to explain in words. Here's the line that we started with (*I draw over his line representing $x + 2y - 4 = 0$*) The point (a, b) is somewhere around, could be anywhere, I could draw it over here or here or here
- 81 Eddie Yeah
- 82 Liz and to find the foot of the perpendicular what I do is to draw this line in (*I draw in the perpendicular from (a, b) to the line*)
- 83 Eddie Oh yeah, yeah, I'm with you
- 84 Liz And the foot of the perpendicular is this point
- 85 Eddie Yeah
- 86 Liz that's what you are trying to find
- 87 Eddie So it's the point where the equation for the normal and the equation for the perpendicular are equal.
- 88 Liz I think .. yes (*laughs*)
- 89 Eddie Well I'll do some working out
- 90 Liz Okay
- 91 Eddie Well in this one we put it in terms of x so $x = 4 - 2y$
- 92 Liz Right
- 93 Eddie And on the second one you just use your equation $y - y_1 = m(x - x_1)$
- 94 Liz okay
- 95 Eddie using a and b as the points which you're given so you've got $y - b = 2$, which is the gradient we've already found out,

- 96 Liz Right
- 97 Eddie $x - a$ - which is $y = 2x - 2a + b$ and as I've got x as the subject of the formula, divide through by 2 which gives us $\frac{y}{2} = x - a + \frac{b}{2}$. Take x to the other side of the bracket then times through by negative 1 which gives me $x = \frac{y}{2} + a - \frac{b}{2}$. So from that I'd use, I've found out $x = 2y$ on the first one so I put $4 - 2y$ is equal to the other side of the x on this, which is $\frac{y}{2} + a - \frac{b}{2}$ times both sides by two to get rid of the fractions, which will give me $8 - 4y = y + 2a - b$ and we'd like to find it in terms of a and b . so first I make b the subject of the formula so $b = 5y + 2a - 8$...
- 98 Liz Yup
- 99 Eddie .. and then make a the subject of the formula, first of all taking over the $2a$ so $2a = b - 5y + 8$ and then divide through by 2 which will give me $a = \frac{1}{2}(b - 5y + 8)$ to eliminate the fractions
- 100 Liz Uhmhm. Okay.
- 101 Eddie It looks a bit messy (*laughs*)
- 102 Liz What is it you've worked out?
- 103 Eddie Er, the coordinates of a and b
- 104 Liz Right, that's um, not strictly what they asked you for.
- 105 Eddie No so I'd have to find the coordinates a intersects, putting x and y as the origins, I think ...
- 106 Liz Putting x and y as ..?
- 107 Eddie As the parameters, like I'd have, make x the subject of the formula as in this case. I've got this equation, say, take this one for example $8 - 4y = y + 2a - b$ just make y the subject of the formula, then find out the y coordinate, then sub it back into the first one which seems the easiest $x = 4 - 2y$ to find the value of x , so just go like this $5y = 8 - b - 2a$
- 108 Liz Yeah, I think it's *plus b*, I don't know which one you were doing it from
- 109 Eddie $5y$ take this over that side, put this over that side you've got $-5y = -b + 2a - 8$, times through by -5 , you've got $5y = b - 2a + 8$ yeah .. so you could divide through by 5 which will give $y = \frac{1}{5}(8 + b - 2a)$

- 110 Liz Uhmhm
- 111 Eddie Then you could sub this in for y in the first one, I don't know if you could write that down, .. so then $x = 4 - 2y$ so it would be $4 - 2(\frac{1}{5}(8 + b - 2a))$, which is just $x = 4 - \frac{2}{5}(8 + b - 2a)$. I'll just work this out to make this easier, so that would leave $4 - \frac{16}{5} + (\frac{2}{5}(b - 2a))$ so you just do .., that's three and one fifth and four minus three and one fifth is four fifths, so $x = \frac{4}{5} + \frac{2b}{5} - \frac{4a}{5}$ so you can take the $\frac{1}{5}$ out again so $x = \frac{1}{5}(4 + 2b - 4a)$
- 112 Liz Okay, so what would you give as the answer to the question?
- 113 Eddie I'd give, er, 'in terms of a and b ...' Well I've found the coordinates, where these cross in terms of x and y but you need to find it in terms of a and b .
- 114 Liz Right, what do you, um, I think you might be mis-interpreting what they mean by 'in terms of'
- 115 Eddie Yeah
- 116 Liz If I said to you that I wanted an answer in terms of a and b what that means is that the answer is allowed to have a and b in it.
- 117 Eddie Yeah, so I'd leave it as it is, x coordinate equals $\frac{1}{5}(4 + 2b - 4a)$ and the y coordinate equals $\frac{1}{5}(8 + b - 2a)$
- 118 Liz Right, so you're quite happy with those answers?
- 119 Eddie Yeah, that 's what I'd leave it as.
- 120 Liz Right, okay. Why is it that we can't give an answer which has got just numbers in it?
- 121 Eddie Because we have no point on the line at all. We're given equations but we're not given any points on the line. I mean for this one (*indicating* $x + 2y - 4 = 0$) you've got ~~the~~ numbers, the equation given at the top of the sheet but for the second one we're given a formula in terms of a and b and not in terms of figures.
- 122 Liz Right
- 123 Eddie I suppose it could be possible to work it out, you could find the point where they intersect, in terms of a number, well I thought you could but probably not (*laughs*)

- 124 Liz Well, it's this point here (*indicating the point (a, b)*) that's the problem isn't it ?
- 125 Eddie Yeah, could be
- 126 Liz I mean, I've drawn it there, but it could have been anywhere
- 127 Eddie Yeah, cos with no given values a and b could be any constant, any variable
- 128 Liz Hmm,
- 129 Eddie So it could be up here, could be over there
- 130 Liz So if you don't know what a and b are there's no way you can say where the foot of the perpendicular's going to be
- 131 Eddie No
- 132 Liz What more information would you need to be able to give numerical answers, answers which are just,
- 133 Eddie Any point on the normal
- 134 Liz Any point on the .. which one do you mean by the normal?
- 135 Eddie Sorry, the perpendicular line, you're given the equation $x + 2y - 4 = 0$
- 136 Liz Yeah
- 137 Eddie Any point on the line perpendicular to that, or the point where they intersect
- 138 Liz Right, okay, I think we'll leave that one. Try that one (*the question reads 'Find the equation of the line with gradient M passing through the point (p, q) '*).....
- 139 Eddie 'Find the equation of the line with gradient M passing through the point (p, q) ' Well again it's the equation $y - y_1 = m(x - x_1)$ We're given the gradient M so it's just $y - q = M(x - p)$. Er you could rearrange that to make any number the constant. You could make M the subject of the formula, x , p , y or q .
- 140 Liz Okay, can you make y the subject then?
- 141 Eddie Yeah so you just $y = Mx - Mp + q$
- 142 Liz Right, um how can you tell what the gradient of that line is?

- 143 Eddie the gradient is $\frac{y - y_1}{x - x_1}$ taking y and x as points on the line, well the first point would be (x, y) the second one would be (x_1, y_1)
- 144 Liz Okay
- 145 Eddie We're not given any, .. well we're given one point (p, q)
- 146 Liz Could you tell me where that line crosses the y -axis?
- 147 Eddie Would it just be where, er, it's the y -axis we're going for?
- 148 Liz yeah
- 149 Eddie Where $Mx - Mp + q = 0$
- 150 Liz Right
- 151 EddieEr if q was equal to zero then it would be when M was equal to ... if the gradient was zero, .. which it wouldn't be because,
- 152 Liz It might be
- 153 Eddie well it could be, which is infinitely inelastic in Economic terms (*laughs*)
- 154 Liz What does that mean?
- 155 Eddie A straight line upwards
- 156 Liz Oh right! Infinitely inelastic?
- 158 Eddie Yeah
- 159 Liz What's the context for that ?
- 160 Eddie It's price and quantity. It just means the percentage change in the y over the percentage change in the x
- 161 Liz And what are x and y usually
- 162 Eddie Er, price and quantity usually
- 163 Liz So it would be circumstances like it doesn't matter how many of them I order it will still be the same price?
- (*A discussion of elasticity in economics and mechanics ensues!*)
- 164 Liz Anyway, we digress, sorry, I was asking you where it would cross the y -axis

- 165 Eddie I've come to the conclusion it would be where
 $Mx - Mp + q = 0$ then I got a bit stuck and started rabbiting on
 about a load of rubbish (*laughs*)
- 166 Liz Um, what I'm interested in is the coordinates of the point
 where it crosses the y -axis
- 167 Eddie Is it where .. just .. I'll divide through by M
 to get rid of the coefficient of x so it would be $M - p + \dots\dots\dots$
 $\frac{q}{M}$ so I take an M outside the bracket and just have $x - p + \frac{q}{M}$..
 then no, hang on, I'll just get rid of that last bit I've just done
 .. take the Mx over the other side, so you've got
 $q - Mp = -Mx$ then times through by negative one so you've
 got $Mp - q = Mx$. To make x the subject of the formula I'd then
 divide through by M so you'd have $p - \frac{q}{M}$ is the x coordinate
- 168 Liz Right
- 169 Eddie And I could also sub. that back in to find the y coordinate
- 170 Liz Uhmhm
- 171 Eddie which would be $y = M(p - \frac{q}{M}) - Mp + q$ just multiply that out so
 you've got $y = Mp - q - Mp + q$, so the qs cancel out , so $y = ..$
 well, zero, 'cos they cancel out as well.
- 172 Liz Uhmhm
- 173 Eddie Yeah ..
- 174 Liz Why have we got zero then
- 175 Eddie Er, because you've got the $Mp - Mp$ which cancels
- 176 Liz Oh, I see they cancel, that result's a bit surprising, isn't it?
- 177 Eddie Yeah, it does, because it's too neat
- 178 Liz (*laughs*) Well think about what I was asking you to find
- 179 Eddie I was trying to find the points where it crossed the x and y axes ..
- 180 Liz The y axis, yeah
- 181 Eddie The y axis .. so that's just justified what I've just said, if that
 subs back in and y is equal to zero then that makes sense.
- 182 Liz Right
- 183 Eddie It's a bit like a proof really that I've done without knowing

- 184 Liz A check that you got the right answer, yeah (*laughs*)
- 185 Eddie I meant it, honest! (*laughs*)
- 186 Liz I can tell that, yeah! Okay that 's great. Um, I think I'll just ask you to do one more because you're probably getting a bit tired of this
- 187 Eddie No' I'm fine (*laughs*)
- 188 Liz So I think we'll go for .. the last one
- 189 Eddie 'Find the equation of the tangent to the curve $y = x^2 + 1$ which passes through the origin' So obviously you've got the point (0, 0) because it passes through the origin.
- 190 Liz Yeah
- 191 Eddie So the tangent - well first I'll find the gradient of the first line, which would be well I could take two points ... on this line, like say when $y = 1$ no let's not take 1, when $y = 5$, x could equal 2 .. or -2
- 192 Liz Uhmhm
- 193 Eddie Er, I'd be better off to differentiate ...
- 194 Liz Uhmhm
- 195 Eddie So, to find the gradient it's $\frac{dy}{dx}$ which is $2x$ because you just disregard the constant, so you just, the gradient would equal $2x$ so when $x = 2$ the gradient would be 4.
- 196 Liz Right
- 197 Eddie So, if that's the gradient of this one, the perpendicular gradient would be $-\frac{1}{2x}$..or $-\frac{1}{2^x}$..
- 198 Liz They're not the same are they? $-\frac{1}{2x}$ isn't the same as $-\frac{1}{2^x}$ If you were writing this one (*pointing to the latter*) just as one fraction it would be $-\frac{x}{2}$
- 199 Eddie Yeah
- 200 Liz So that 's the simplest form you can get really (*pointing to the former*)

- 201 Eddie Yeah, so I just get $-\frac{1}{2x}$ so it would be $y - 0$ (which is just y)
 $= -\frac{1}{2x}$ (which is the gradient) times $(x - 0)$ (which again is x) so
 you've got $y = -\frac{x}{2x}$.. so then you can take it out as $y = -\frac{1}{2}$
- 202 Liz Uhmhm
- 203 Eddie (*very quietly*) 'Find the tangent' So should the equation just
 be $y = -\frac{1}{2}$?

Break in tape of a few seconds

- 204 Eddie It would just be a straight line wouldn't it? Is that working?
 (*referring to the tape recorder*)
- 205 Liz It is - it should be auto-reversing - I'd better just make sure it is.
 It's got to the end of the tape - it's only a D60 you see. It's not
 displaying anything. I have a feeling that the battery may be
 running down. It's probably recording okay. I haven't got any
 more batteries with me anyway so we'll go on and see if
 anything comes out. Um
- 206 Eddie So that's what I was saying it's a straight line
- 207 Liz Yeah - that would be a horizontal line
- 208 Eddie Yeah - a horizontal line where $y = -\frac{1}{2}$ at all points.
- 209 Liz Right - and that wouldn't go through the origin
- 210 Eddie No
- 211 Liz Try doing a little diagram of what the tangent at the point that
 we're talking about would look like and see if that helps
- 212 Eddie You've got your origin $(0, 0)$, you've got the $y = x^2 + 1$, so that
 $x = 0$, $y = 1$ so it would just be that line curving upwards like
 that - that point 1 so that's that one - so the inverse would just
 be - no the .. , what's the word? - the normal would just be the
 opposite to that - it would cut there (*during this speech Eddie
 draws a sketch of $y = x^2 + 1$*)
- 213 Liz We're not really concerned with the normal
- 214 Eddie 'the tangent to the curve' - so it would just be the tangent to
 that line

- 215 Liz Well - it's a tangent to that curve, which means it's got to brush past, just touch that curve somewhere and it goes through the origin
- 216 Eddie The thing that's putting me off is when $y = 0$, x would have to equal the root of -1 .
- 217 Liz Right - on that equation, if you put $y = 0$ then you're asking 'where does it cross the x -axis?'
- 218 Eddie Yeah
- 219 Liz And it doesn't does it?
- 220 Eddie No, not at all
- 221 Liz So that's why you get the square root of -1
- 222 Eddie Yeah
- 223 Liz It doesn't exist
- 224 Eddie Without complex numbers
- 225 Liz That's P2! (*laughs*)
- 226 Eddie I haven't done it anyway! (*laughs*) So obviously it crosses through this point here (*indicating the origin*) I think it will just brush along the edge of that curve there (*he sketches in the tangent*) so you'd find the gradient at that point where they touch
- 227 Liz Right
- 228 Eddie And that would be equal to that one So points where they cross are points on the $y = x^2 + 1$ graph .. and the gradient will be equal at that point as to the line .. (*coughs*) that goes through the origin
- 229 Liz You've got to talk for an hour next session as well, you know
- 230 Eddie Yeah, I know I have
- 231 Liz you won't have any voice left
- 232 Eddie I'll just have last session off! Would I get any marks for this so far?
- 233 Liz (*laughs*) I'm not answering that question

- 234 Eddie Well obviously the gradients are the same at the point (a, b) (by now he has marked the point of intersection of curve and tangent as (a, b)) on that (referring to the curve) as to the normal there (pointing to the tangent). And the gradient of that line is $2x$ so at the point a the gradient of the line is $2a$
- 235 Liz Okay
- 236 Eddie (*murmuring*) at that point there - so the gradient would have to be the same there. And you'd just do something with (a, b) 's - put them into a formula. Might not help but it might, so I'll have a go. So $y - b = 2a$ (which is the gradient) times $(a - b)$ oh no it's $(x - a)$ No I can't get anywhere.
- 237 Liz You're doing fine
- 238 Eddie Am I?
- 239 Liz Yeah, you've made a really good start there. There's two pieces of information that you're not using - yet
- 240 Eddie The origin
- 241 Liz Yeah
- 242 Eddie And the equation of that line (*pointing to the curve*)
- 243 Liz Uhmhm
- 244 Eddie But how? Er I've got an idea - I'll try it - so that would be $2ax - 2a^2 + b = x^2 + 1$ at that point there
- 245 Liz Yeah
- 246 Eddie So then in terms of a and b I've got to find that coordinate there which will be on both lines
- 247 Liz Right
- 248 Eddie And then onwards it would be pretty straightforward to find the equation of the line, but this is the hard bit. So you've got $x^2 + 1 = (-2a)^2 + (2ax + b)$ So I could be wrong here, but I think $-2a$ would be equal to x as they're both the coefficients of x^2 .. it's a bit - it doesn't sound right
- 249 Liz No it's not - they're both the coefficient of x^2 you said - well they're not
- 250 Eddie They're the coefficient of the squared term
- 251 Liz Yeah - but that doesn't necessarily mean they're equal because it's not the same squared term ...

- 252 Eddie But what I thought was - you'd have your x in brackets there, squared plus one, that would be equal to that in brackets squared and this would be equal to one
- 253 Liz No, not necessarily
- 254 Eddie It looked nice though (*laughs*)
- 255 Liz It's a way out of a sticky problem!
- 256 Eddie I could just make a and b the subjects of the formulas in terms of x which would be hopefully a move in the right direction .. so you just have, well, start with the b because it's a bit simpler - so $b = x^2 + 1 + 2a^2 - 2ax$ could you break that down at all? So $b =$ it won't work
- 257 Liz You're heading a bit off in the wrong direction and we haven't got all that much time left, so I'll
- 258 Eddie give me some help
- 259 Liz For the purposes of my research it doesn't really matter whether you finish this question or not, but you'd probably feel better about it if you did
- 260 Eddie Yeah, I would
- 261 Liz So I'll steer you back on course. Um, you see that equation you've got written at the bottom of that page (*I am referring to* $x^2 + 1 = (-2a)^2 + (2ax + b)$) um, think about why you put that equation together
- 262 Eddie I put that equation down because they're both in terms of y so y on that line there is equal to $2ax - 2a^2 + b$ and y there is equal to $x^2 + 1$ as well so they both touch
- 263 Liz So it's actually the sort of thing you would do if you wanted to find the coordinates of the point where the curve and the line touch
- 264 Eddie Yeah
- 265 Liz But in fact you know what the coordinates are - or you've made it up, you've called them a and b
- 266 Eddie Yeah
- 267 Liz and you've got your made-up letters in this equation anyway
- 268 Eddie Yeah

- 269 Liz so there's not much point in forming an equation to try and find out what those two are, when you already know they're a and b
- 270 Eddie Yeah
- 271 Liz So we're actually trying to look at it from another angle, which is if I put a in for x and b in for y then the, that equation and the equation that you've just worked out here for the tangent should both be true at the same point
- 272 Eddie So it would be where the problem was working out the gradient there because I think the, I think $2x$ is right, so it would be this - the gradient - you've got $y = x^2 + 1$ and the gradient for the other line would be $y - y_1 = m(x - x_1)$
- 273 Liz You've done that
- 274 Eddie Yeah - I'm just sort of putting it in perspective, then b would equal $\frac{1}{2x}$ (which is the gradient) times a (which is that minus that (referring to $x - x_1$ in $y - y_1 = m(x - x_1)$) because they're both the origin.
- 275 Liz Hmm
- 276 Eddie So
- 277 Liz I'm sorry, I can't see where you got that from
- 278 Eddie That's where you've got $y - y_1$ which is $b - 0$ equals $\frac{1}{2x}$ which is the gradient I worked out before
- 279 Liz Yeah - it wasn't $\frac{1}{2x}$ was it?
- 280 Eddie $-\frac{1}{2x}$
- 281 Liz No - you differentiated and you got $2x$
- 282 Eddie Yeah - that would be the gradient at any point
- 283 Liz Any point on the curve
- 284 Eddie Ah, so it would be $\frac{1}{2a}$ because that's taking the coordinates there as a , the x -coordinate.
- 285 Liz Okay

- 286 Eddie So the gradient would be $-\frac{1}{2a}$
- 287 Liz All right - it wouldn't actually
- 288 Eddie It wouldn't?
- 289 Liz Because you got the gradient as this (*pointing to $2x$*) but you then got the $-\frac{1}{2x}$ when talking about the normal, and the normal doesn't actually come into it
- 290 Eddie Yeah
- 291 Liz It's the gradient of the curve we're interested in -
- 292 Eddie Ah
- 293 Liz the gradient of the tangent
- 294 Eddie Yeah, yeah
- 295 Liz So it's not $-\frac{1}{2x}$
- 296 Eddie So it's where the coordinates of the point are equal so it would still be $2x$ because it's where that line there - the gradient is equal to that line there
- 297 Liz Yeah
- 298 Eddie the gradient of that would be $2x$ as well ... so that would be where $b = 2ax$... $\cos 2x$ would be the gradient, which would be m times x as the other coordinate is $(0, 0)$ so that would be $b = 2ax$...
- 299 Liz Uhmhm
- 300 Eddie You were thinking about replacing that x with an a a minute ago
- 301 Eddie Yeah because a would be that coordinate there so I'd say $b = 2a$, because a would be the coordinate there - wouldn't it - and $2x$ would be the gradient - so that would be $2a$
- 302 Liz Well
- 303 Eddie The gradient would be equal to $2a$
- 304 Liz Yes
- 305 Eddie So $2a$ would be equal to $2x$

Brief interruption from another student coming into the room

- 306 Eddie Is it fair to say that $2a$ would be equal to $2x$ in that situation?
- 307 Liz Yeah .. so if you replace the x by a
- 308 Eddie You'd have $2a^2$
- 309 Liz Yeah
- 310 Eddie So say that's, say $2a^2 = b$ would be equal - that's one equation and the other (*inaudible*) so b is the y coordinate, so if I took b to the other side you could have $2a^2 = x^2 + 1$... so say that's $2a^2 = b$ and b is the y coordinate - it would be equal to $x^2 + 1 = y$ so b and y are the same thing
- 311 Liz Uhmhm
- 312 Eddie So that's $2a^2 = x^2 + 1$
- 313 Liz Right - but also at that point x and a are the same thing
- 314 Eddie Yeah - so it would just be so you'd get one - you'd get the x coordinate, $x = \dots$ that would be minus actually - you'd just have $x^2 = 1$ so x would equal 1
- 315 Liz yeah or -1
- 316 Eddie Oh yeah - or -1 and then sub that in to find the y , so that's just $2x^2$ so y would be equal to 2
- 317 Liz What have you found there?
- 318 Eddie I've found the point (a, b) so $a = \pm 1$ and $b = 2$
- 319 Liz Okay
- 320 Eddie So then from that you can just work out the $y - y_1$ which is $2 - 0$ equals - the gradient's $2x$ which is 2 times ± 1 which will be ± 2 times $x - x_1$ which will be $\pm 1 - 0$ Mmm
- 321 Liz (*laughs*) That seems a lot of work to prove that $2 = 2$
- 322 Eddie Yeah - wouldn't that prove that it has to be a positive value on the x ?
- 323 Liz No - because you see you've got ± 2 here
- 324 Eddie Yeah
- 325 Liz - and ± 1
- 326 Eddie Yeah so it's (*inaudible*)

- 327 Liz - so it's +2 times +1 or -2 times -1 and they're both right
- 328 Eddie Yeah
- 329 Liz Um - once you got to here you told me that you'd found a and b
- 330 Eddie Yeah
- 331 Liz Right - now what were you trying to do in the next stage?
- 332 Eddie I was trying to find out the equation of that line because you've got the $x_1 - x_2$ and $y_1 - y_2$ which would be $2 - 0$ equals the gradient which is m times $a - 0$ and a is given as ± 1
- 333 Liz Okay - that's ..
- 334 Eddie the gradient ..
- 335 Liz The working that you've done hasn't ended you with the equation of a line has it?
- 336 Eddie No
- 337 Liz What's missing?
- 338 Eddie x and y
- 339 Liz M m m
- 340 Eddie So I'll take the x and y as x and y and I'll just use the ± 1 and ± 2 as the second coordinates - $y - 2$ equals the gradient which is $2a$ or $2x$...would it be fair to assume that it's a positive one as I've drawn it there - so it can be a negative one which is across there? There's a logic in that (*laughs*) x equals +1 so that would be $2(x - 1)$ so then $y = 2x$ that's just it because the negative 2s cancel out - so the equation of the tangent will be $y = 2x$
- 341 Liz Marvellous
- 342 Eddie Took long enough though!
- 343 Liz It's a very difficult question
- 344 Eddie Mmm - cos it looks so small and straightforward but it's not
- 345 Liz Yeah - well it - um, normally when you're asked to find the equation of a tangent you're told that it's a tangent to this curve at this point
- 346 Eddie yeah

- 347 Liz - so you can differentiate and then you can find out as a number what the gradient of the tangent is - but on this one you couldn't because you didn't know which point the tangent was at - you had to actually find out what this point was
- 348 Eddie yeah
- 349 Liz before you could do it - that was what made it hard, but
- 350 Eddie It worked eventually (*laughs*)
- 351 Liz yeah (*laughs*)
- 352 Eddie after a lot of assistance
- 353 Liz that was a very good method - I think you did pretty well on that.

Frank 14/6/94

- 1 Liz: Right. Umm, these don't come in any particular order. .. Try this one first. (*The question reads 'Sketch $y = x(x - a)$ '*)....There's a pen if you want one. Anything else you want. Have you got any initial reactions to the question?
- 2 Frank: Umm, well yeh. Sketching a graph with no constant is sketching graphs to start with I'm not too keen on that.
- 3 Liz: Uhmhm.
- 4 Frank: Sketching a graph without a constant to get you started off.
- 5 Liz: Right.
- 6 Frank: Umm I think I'd .. expand that.
- 7 Liz: Uhmhm
- 8 Frank: So it's better (*writes $y = x^2 - ax$*) Well I know it's a parabola
- 9 Liz: right
- 10 Frank: and it's a positive one.
- 11 Liz: Uhmhm.
- 12 Frank: I'm just trying to work out where the axes are ..
- 13 Liz: Right It's a pity you weren't in the lesson last week because we were doing a bit of this,
- 14 Frank: right, right
- 15 Liz: and then you would have been reminded about it. Umm, to find out where the axes go, that's what you're up to now
- 16 Frank: yes
- 17 Liz: umm what you're asking there really is where does the curve cross the axes?
- 18 Frank: Yes.
- 19 Liz: Where does it cross the y -axis? Where does it cross the x -axis? Umm and the answer to those questions comes from putting $x = 0$, to find out where it crosses the y -axis. And $y = 0$ to find out where it crosses the x -axis.
- 20 Frank: Right, $x = 0$ will be, ... you get nothing.
- 21 Liz: Uhmhm.

- 22 Frank: So it's at the origin. Umm well $y = 0$ is as well isn't it? (*Frank has already drawn a parabola and now begins to draw in the y-axis as a tangent at the turning point*)
- 23 Liz: So it definitely goes through the origin
- 24 Frank: yes
- 25 Liz: umm but we can't be sure which point on the curve
- 26 Frank: it is
- 27 Liz: it is, and it might be that the point you were suggesting is that it's at the bottom, at the turning point.
- 28 Frank: Yes it could be there
- 29 Liz: yes
- 30 Frank: or anywhere.
- 31 Liz: Right.
- 32 Frank: It's the a that's throwing me.
- 33 Liz: Hmm, tell me about that.
- 34 Frank: If I had a number to work around then I'd have a starting point to look at what the x was
- 35 Liz: right
- 36 Frank: but without that, I've got to, because it could be absolutely anything.
- 37 Liz: Uhmhm. Okay, well umm, let's see if we can make some progress by replacing a by a number then.
- 38 Frank: Okay.
- 39 Liz: And, as you say it could be absolutely anything so it doesn't matter which number we choose,
- 40 Frank: Uhmhm
- 41 Liz: so try putting $a = 2$.
- 42 Frank: y equals $x^2 - 2x$, and I've got to get that to equal 0 to find where it crosses.
- 43 Liz: Uhmhm.
- 44 Frank: It would have to be 2. 2 would work anyway because then it'd be 2 times 2, 2 times 2.

- 45 Liz: Uhmhm.
- 46 Frank: So it cross ... equals 2, it would be more, more there than there. (*He adds the axes to the diagram in appropriate places*)
- 47 Liz: Uhmhm.
- 48 Frank: As opposed to there. (*He points to the original positioning of the y-axis*)
- 49 Liz: Right, okay, so what's this point here? (*I point to the second point of intersection between the curve and the x-axis*)
- 50 Frank: This?
- 51 Liz: Hmm.
- 52 Frank: It's, I've forgotten so much about curve sketching. Umm I've tried to shut maths out
- 53 Liz: right
- 54 Frank: to tell you the truth
- 55 Liz: h m m
- 56 Frank: Apart from statistics.
- 57 Liz: Ah ha. Well that's the place where the curve crosses the x-axis isn't it?
- 58 Frank: Yes.
- 59 Liz: Umm which is what you were trying to find out when you were doing $y = 0$.
- 60 Frank: Hmm.
- 61 Liz: So that would be the point (2, 0).
- 62 Frank: Yes.
- 63 Liz: $x = 2$. Okay, now, what about this business that it was an a not a 2.
- 64 Frank: Uhmhm.
- 65 Liz: You've got a sketch there of $y = x^2 - 2x$. (*the emphasis is on 2*)
- 66 Frank: Uhmhm.
- 67 Liz: Umm, what difference would the a make?
- 68 Frank: That a could move it absolutely anywhere.

- 69 Liz: Yes, what would it move?
- 70 Frank: It'd move the umm the parabola around.
- 71 Liz: Uhmhm.
- 72 Frank: It, it made it umm on the y -axis not the x -axis because you've got that x more or less.
- 73 Liz: Umm what do you mean by that exactly?
- 74 Frank: Umm the way we found it out when x was nought,
- 75 Liz: Uhmhm
- 76 Frank: we know that when x is nothing, no matter what a is it's always gonna be there (*pointing to the origin*), yes.
- 77 Liz: Right.
- 78 Frank: So it will always be in the same place. It's going to be (*inaudible*).
- 79 Liz: Umm, when $x = 0$, $y = 0$
- 80 Frank: yes
- 81 Liz: and that doesn't depend on what a is, that's always true.
- 82 Frank: Yes.
- 83 Liz: So the fact that $x = 0$ and $y = 0$, what does that tell you about the curve? Which feature on your diagram is that an interpretation of? What, what did you do with the information $x = 0$, $y = 0$?
- 84 Frank: I found out where it first crossed and then where the origin was on the sketch.
- 85 Liz: You found out that the origin was on the curve?
- 86 Frank: Yes.
- 87 Liz: All right, so the fact that the curve has to go through the origin doesn't depend on what a is.
- 88 Frank: No.
- 89 Liz: All right, okay. So umm what are, what does change if you change a ?
- 90 Frank: Where it is on .. on the origin. Which part of the curve it is, whether it's the vertex of a curve, how far up it is, how far across it is.

- 91 Liz: Right, okay. But it's always going to be a parabola.
- 92 Frank: Yes.
- 93 Liz: So umm that sketch that you've done there, you haven't actually labelled
- 94 Frank: Uhmhm
- 95 Liz: this point (*I point to the second intersection between the curve and the x-axis*). I can see *that's* the origin.
- 96 Frank: Yes.
- 97 Liz: But you haven't actually labelled *that* point. Umm in fact as long as a is positive
- 98 Frank: Uhmhm
- 99 Liz: that diagram could represent any of these graphs,
- 100 Frank: right
- 101 Liz: $x^2 - ax$
- 102 Frank: Uhmhm
- 103 Liz: because the origin must be to the left of where it, the other place where it crosses.
- 104 Frank: Yes.
- 105 Liz: If umm I'd asked you to put in a equals let's say -3 ,
- 106 Frank: it would be the other side.
- 107 Liz: It would be the other side.
- 108 Frank: Right.
- 109 Liz: So even with something as vague as $y = x^2 - ax$,
- 110 Frank: Uhmhm
- 111 Liz: which as you say has got erh another variable in it
- 112 Frank: yes
- 113 Liz: and you don't know what it is, even with some things as vague as that, umm you can draw a pretty good sketch.
- 114 Frank: Yes.
- 115 Liz: There's really only two things, two shapes that it could look like.

- 116 Frank: Yes.
- 117 Liz: Okay. Fine. You've done well on that. Umm try that one.
(Factorise $x^3 - 1$, $x^3 - 8$ and $x^3 - a^3$) Now this is erh, this is going back a bit actually.
- 118 Frank: It is.
- 119 Liz: You actually did this umm sometime before Christmas and we were doing it just after you'd learned about the factor theorem.
- 120 Frank: Uhmhm.
- 121 Liz: Now does that term mean anything to you or should I remind you what it was
- 122 Frank: The factor theorem's to do with erh long division isn't it?
- 123 Liz: Uhmhm.
- 124 Frank: It's, I'm trying to remember what it's used for.
- 125 Liz: Well it's mainly used as a way of finding one factor of an expression like umm, like those or
- 126 Frank: Uhmhm
- 127 Liz: quartics, quintics, things with x^4 , x^5
- 128 Frank: yes
- 129 Liz: or even worse Umm and what the factor theorem says is if you can find a number which makes that expression zero, when you substitute it in, then you've found a factor of it. Say if umm you substituted 3 for x , and the expression comes to zero.
- 130 Frank: Hmm.
- 131 Liz: And then that means that $x - 3$ is a factor.
- 132 Frank: Yes.
- 133 Liz: So, if you're trying to factorise say the first one of those
- 134 Frank: yes
- 135 Liz: the first thing you'll want to do is to find a number which makes it come to zero.
- 136 Frank: Uhmhm.
- 137 Liz: And then once you've done that you've found one factor.
- 138 Frank: Yes. You're first one would be -1.

139 Liz: Umm -1 ?

140 Frank: -1^3 is -1 .

141 Liz: Uhmhm.

142 Frank: Take away -1 gives nothing (*writes $-1 - 1$*).

143 Liz: Erh it's not take away -1 .

144 Frank: It's take away 1 .

145 Liz: Yep.

146 Frank: This minus take away 1 If I do it the other way round it would be minus, it would be 1 take away -1 (*writes $1 - -1$*).

147 Liz: What do you mean by doing it the other way round?

148 Frank: Instead of being -1 take away 1

149 Liz: Uhmhm

150 Frank: 1 take away -1 .

151 Liz: Hmm.

152 Frank: Yes?

153 Liz: Those two aren't the same (*points to $-1 - 1$ and $1 - -1$*)
..... If you substitute -1 for x in that first expression

154 Frank: h m m

155 Liz: which one of those two do you get?

156 Frank: You mean that one? (*points to $x^3 - 1$*)

157 Liz: Yes.

158 Frank: Which equals -2 .

159 Liz: That's right.

160 Frank: You need that x to equal 1 .

161 Liz: Uhmhm. That's right.

162 Frank: Yes.

163 Liz: So that means that $x - 1$ is a factor.

164 Frank: Yes.

- 165 Liz: And then one way of finding what the other factor is to do long division.
- 166 Frank: Hmm.
- 167 Liz: Can you remember how to do that.
- 168 Frank: Umm I think so. Do I use that (*points to $x - 1$*) umm to divide it by?
- 169 Liz: Yes, that's right.
- 170 Frank: Right.
- 171 Liz: When you umm write down your $x^3 - 1$, you'll probably find it better to put
- 172 Frank: the rest of the x s in (*starts to set out a long division calculation*)
- 173 Liz: some yes, so you've got no x^2 's, and no x s. *minus 1* at the end.
- 174 Frank: Oh dear. Right. I can't remember how to do it
.....
- 175 Liz: You umm deal with the first terms to start with
- 176 Frank: yes
- 177 Liz: so you're saying, we're looking at the x and the x^3 . (*points to these terms*)
- 178 Frank: Yes.
- 179 Liz: And you're saying 'what do I need to multiply x by to get x^3 '?
- 180 Frank: Uhmhm.
- 181 Liz: Which is x^2 , so x^2 is like your first answer. So if you put, are you putting your answers up here and then working down below, usual sort of pattern
- 182 Frank: yes
- 183 Liz: so you'd put x^2 up here above the x^3 , okay
- 184 Frank: Uhmhm
- 185 Liz: so we're saying that $x - 1$ goes into this x^2 times,
- 186 Frank: yes

- 187 Liz: umm and then you have to work out what the remainder is so that you can like carry it into the next column.
- 188 Frank: Right.
- 189 Liz: And to do that, you say 'well what do I get if I multiply $x - 1$ by x^2 ?'
- 190 Frank: We get x . So it goes there?
- 191 Liz: umm you'll get x^3 first, won't you?
- 192 Frank: Yes.
- 193 Liz: That goes underneath the x^3 . And then you'll get umm $-x^2$, we're doing the x^2 times -1
- 194 Frank: yes
- 195 Liz: so that goes next to it.
- 196 Frank: Right.
- 197 Liz: Umm and then you subtract that that you've just worked out from the line above to see what you're going to
- 198 Frank: so that subtract that, yes?
- 199 Liz: Yes.
- 200 Frank: I think umm, $-x^2$ yes?
- 201 Liz: Umm you're subtracting the bottom line from the top line
- 202 Frank: oh yes
- 203 Liz: so you're now actually subtracting $-x^2$, which means it comes out as $+x^2$.
- 204 Frank: Yes.
- 205 Liz: ... Okay, and then you would bring down the next term
- 206 Frank: Uhmhm
- 207 Liz: which is nothing in this case.
- 208 Frank: Okay.
- 209 Liz: No x s, umm and then see how many times this goes into that, so we're looking at x into x^2
- 210 Frank: right, so it just, that comes down here yes?
- 211 Liz: Yes.

- 212 Frank: so that's no x , so x goes into no x
- 213 Liz: No, into this one (*points to x^2*).
- 214 Frank: That.
- 215 Liz: Uhmhm.
- 216 Frank: Need erh x^2 you get x , (*writes x in the 'answer line'*)
- 217 Liz: Uhmhm.
- 218 Frank: And x^2 again (*writes x^2 in the working below*)
- 219 Liz: yes
- 220 Frank: minus x .
- 221 Liz: Uhmhm.
- 222 Frank: Umm .. that's nothing .. and that's x (*writing in the answers to the subtraction*)
- 223 Liz: ah ha
- 224 Frank: it would mean that that would be it
- 225 Liz: Right.
- 226 Frank: -1 But umm (*inaudible*)
- 227 Liz: ah hmm, well no, this one then.
- 228 Frank: Do this one.
- 229 Liz: Yes.
- 230 Frank: x into x is nothing
- 231 Liz: 1
- 232 Frank: 1. x and that's nothing, .. and that's .. $-1 + 1$.. do I times them or?
- 233 Liz: No umm what, when you've got this, this line here, this line here
- 234 Frank: yes
- 235 Liz: and this line here, what you're doing is timesing the $x - 1$ by whatever you've just written on this line.
- 236 Frank: Yes.
- 237 Liz: So we're doing $x - 1$ times 1, which will be just $x - 1$.

- 238 Frank: Yes. that'll be -1 take away -1 which is nothing
- 239 Liz: Umm. So you've finished and there was no remainder
- 240 Frank: yes
- 241 Liz: which is what you were expecting because you think $x - 1$ goes in exactly.
- 242 Frank: Yes.
- 243 Liz: So that top line is your answer.
- 244 Frank: Yes.
- 245 Liz: Okay, and so the instruction is to factorise $x^3 - 1$
- 246 Frank: Uhmhm
- 247 Liz: so that your complete answer is that $x^3 - 1 = (x - 1)(x^2 + x + 1)$.
- 248 Frank: $x - 1$ goes in there, that's one of them yes? (*He is trying to factorise $x^2 + x + 1$*)
- 249 Liz: Yes.
- 250 Frank: And $x - 1$ as well.
- 251 Liz: Umm, where are you getting the second $x - 1$ from?
- 252 Frank: To get that you times the two figures together
- 253 Liz: right
- 254 Frank: yes, then -1 times -1 is 1
- 255 Liz: h m m
- 256 Frank: $-x + -x$, if it was -1 that x would be wrong wouldn't it?
- 257 Liz: Hmm. Yes, I'm going to save you a bit of time here. That quadratic, $x^2 + x + 1$ it doesn't actually factorise.
- 258 Frank: Right.
- 259 Liz: So there's nothing more you can do with that.
- 260 Frank: Okay.
- 261 Liz: So you're left just with saying that $x^3 - 1$, the thing we had to factorise originally, is $(x - 1)(x^2 + x + 1)$.
- 262 Frank: Hmm.

- 263 Liz: Those are the two factors. You can't split them up any more because that one doesn't factorise. Okay. Umm, what about $x^3 - 8$?
- 264 Frank: 2 would go in, so $x - 2$ would be the first factor.
- 265 Liz: Yes.
- 266 Frank: You have to go through this again to find out what that one is, yes?
- 267 Liz: Right. (Frank begins writing the working for dividing $x^3 - 8$ by $x - 2$. He writes x^2 where he should have x^3) That's x^3 there.
- 268 Frank: Yes.
- 269 Liz: What you've just written.
- 270 Frank: Yes.
- 271 Liz: You're taking the $x - 2$ and timesing it by x^2 .
- 272 Frank: It's the second part that I'm not happy with. (inaudible)
- 273 Liz: Right, well, what you're doing there is you're taking this, these both of these terms and timesing by that, so you timesed x by x^2 and go that
- 274 Frank: It'd be $-2x^2$
- 275 Liz: that's right
- 276 Frank: (He starts doing the subtraction) that would be, that would be .. erh .. plus $2x^2$
- 277 Liz: Uhmhm
- 278 Frank: Then that term comes down to here (referring to the $0x$ term)
- 279 Liz: Yes.
- 280 Frank: That's ... $2x$ yes? (writing $+ 2x$ in the top line)
- 281 Liz: Right.
- 282 Frank: Is that -4 ? (pointing to the space next to $2x^2$ in the bottom line)
- 283 Liz: $-4x$ yes.

- 284 Frank: (He writes $0 + 4x$ in the next line of the calculation)..... $+ 4$, not $+ 4x$ (He crosses out the x)
- 285 Liz: Why's that?
- 286 Frank: No it is $+ 4x$
- 287 Liz: yes, you're taking $- 4x$ from nothing.
- 288 Frank: Yes. I was looking at it as if there's an x there (*indicating the $0x$ two lines above*)
- 289 Liz: right
- 290 Frank: not as if there was no x .
- 291 Liz: Yes, yes.
- 292 Frank: that comes down (*writing $- 8$ at the end of the bottom line of the calculation*) Now I've lost myself
- 293 Liz: Umm, now you're seeing how many times this $x - 2$ goes into
- 294 Frank: yes
- 295 Liz: $4x$.
- 296 Frank: Right. It goes into there $4 \dots$ minus (*inaudible*) Is that right?
- 297 Liz: Uhmhm.
- 298 Frank: So (*inaudible*)
- 299 Liz: Yes. Okay, umm and again you've got a quadratic that you can't factorise there.
- 300 Frank: (*He writes $(x - 2)(x^2 + 2x + 4)$*) There.
- 301 Liz: Right. What about the third one?
- 302 Frank: The third one? Erh x would equal a so it would be $x - a$ there.
- 303 Liz: Uhmhm. Do you want another piece of paper so you can look at the first side at the same time?
- 304 Frank: Okay, yes. (*He begins setting out the working for a long division but puts $+ a^3$ at the end of the dividend*)
- 305 Liz: It's $- a^3$ at the end (*Frank begins doing the long division*)
- 306 Frank: (*inaudible*) So they're the two bits there. (*He has written $(x - a)(x^2 + ax + a^2)$*)

- 307 Liz: Right. Okay. Can you say anything about that third one in relation to the first two.
- 308 Frank: ... I found it easier.
- 309 Liz: Uhmhm.
- 310 Frank: I think that might have just been the practice, I'm not sure.
- 311 Liz: Yes.
- 312 Frank: Umm, even though you're dealing like with two variables and it is just the same, just as, exactly the same method.
- 313 Liz: right
- 314 Frank: there's no change.
- 315 Liz: Okay. Supposing I asked you to do $x^3 - 27$.
- 316 Frank: Hmm.
- 317 Liz: What would you do?
- 318 Frank: Umm it would be $x - 3$ and then I'd lay it out just like that and $x^3 + 0x^2 + 0x - 27$.
- 319 Liz: Right, yes.
- 320 Frank: and then I'd just work through it.
- 321 Liz: Okay. Umm, this third one that you've done
- 322 Frank: Uhmhm.
- 323 Liz: umm has got an a in it instead of a number.
- 324 Frank: Yes.
- 325 Liz: Umm, what's the point of that?
- 326 Frank: Because the general rules so you can see how it works for any number at all.
- 327 Liz: Right. So you can see how it works for 3 for instance.
- 328 Frank: Yes.
- 329 Liz: Can you predict what the answer would be if you did the division?
- 330 Frank: Umm it would be $x^2 + 3x + 9$?
- 331 Liz: So in fact having done that one if I asked you to factorise $x^3 - 27$ you wouldn't need to do the rest of the work.

332 Frank: Yes, you just put the numbers into that.

333 Liz: Yes. So if you can deal with something like this one, straightaway,

334 Frank: yes

335 Liz: umm then that's the only one you need to do.

336 Frank: Yes.

337 Liz: The rest of them fit the same pattern.

338 Frank: Uhmhm..

339 Liz: Okay, I think you're getting warmed up a bit now. Sorry I'm putting you through this purgatory.

340 Frank: That's all right

341 Liz: Last one.

342 Frank: Last one?

343 Liz: Yes.

344 Frank: (*Reads*) 'Find the equation of the line with gradient M which passes through the point (p, q) ' Okay Let's try it with $y = \dots mx^c?$

345 Liz: $mx + c$.

346 Frank: $+ c$. (*writes $y = mx + c$*)

347 Liz: What's that formula about?

348 Frank: That's the general formula for a linear equation to

349 Liz: Uhmhm, equation of a straight line.

350 Frank: Yes.

351 Liz: Right. What do the y , m , x and c mean?

352 Frank: Umm well y and x are the axes,

353 Liz: Uhmhm.

354 Frank: m is the gradient and c is the constant .. of the .

355 Liz: Yes, what do you mean by the constant?

356 Frank: Like umm the first one we did had a as the constant.

357 Liz: Right, yes.

358 Frank: That's what the c would be

359 Liz: okay

360 Frank: here. Umm I'm trying to remember how to find c It's $x_1 - x_2$ or something like that.

361 Liz: Hmm.

362 Frank: I can't remember how to erh ... I've only got one set of points, so the equation could actually be, it could have *any* gradient.

363 Liz: .. Well it says it's got gradient M

364 Frank: Yes. Hmm. I can't see how that ..

365 Liz: What information, what more information than you've got there

366 Frank: hmm

367 Liz: would you need to be able to draw the line? If I gave you a sheet of graph paper and you could plot it anyhow you like?

368 Frank: ... Where do they actually cross the axes

369 Liz: Both of them.

370 Frank: I think. Yes.

371 Liz: Okay.

372 Frank: Or at least to have an actual figure for the gradient rather than M .

373 Liz: Right.

374 Frank: If M wasn't just a variable.

375 Liz: So suppose I say that umm M is four.

376 Frank: Uhmhm..

377 Liz: What else do you want to know?

378 Frank: Umm if I knew where it crossed either of the axes I could work out

379 Liz: Uhmhm.

380 Frank: where the line was.

381 Liz: Right. I'm not very keen to tell you where it crosses

382 Frank: yes

- 383 Liz: either of the axes. Is there any other information that would be acceptable?
- 384 Frank: Umm what p and q are.
- 385 Liz: What p and q are, okay. So if I told you a value for M
- 386 Frank: ah hmm
- 387 Liz: and I told you a value for p and a value for q
- 388 Frank: ah hmm
- 389 Liz: then you'd know how to do it.
- 390 Frank: U m m
- 391 Liz: apart from that, let's change the question. Then you'd be able to draw it.
- 392 Frank: Yes.
- 393 Liz: Okay. Umm well suppose M is 4, p and q are umm 2 and 3.
- 394 Frank: Okay. Erh excuse the graph (*He draws a rough sketch showing the point (2, 3) and a line of gradient 4 passing through it, The point (1, -1) is also shown as being on the line*) it sort of goes umm ... yes
- 395 Liz: Uhmhm.. That should really be a straight line and you're not particularly to scale, so
- 396 Frank: I know. It isn't.
- 397 Liz: You've got the idea.
- 398 Frank: Yes.
- 399 Liz: Okay. Umm now how would you find the equation of that line?
- 400 Frank: Erh it well from that I made that another point on the line is $(-1, 1)$.. $(1, -1)$
- 401 Liz: Uhmhm.
- 402 Frank: yes, so I've got umm $(2, 3)$ and $(-1, ..$ and $(1, -1)$
- 403 Liz: yes
- 404 Frank: then I've got the gradient's four, I'm just trying to remember how you actually get to that equation.
- 405 Liz: Hmm. Well let me show you how you'd do it for this case that I've chosen the numbers for.

406 Frank: Uhmhm..
407 Liz: Umm you told me that in the equation $y = mx + c$
408 Frank: yes
409 Liz: and the m is for, is the gradient.
410 Frank: Yes.
411 Liz: So for this line it's going to be 4.
412 Frank: Yes.
413 Liz: So all that remains is to find out what c is.
414 Frank: Yes.
415 Liz: Umm now we can do that by using the coordinates for any point that we know is on the line
416 Frank: yes
417 Liz: because any point that actually lies on the line will have coordinates which satisfy the equation
418 Frank: yes
419 Liz: they make the equation work. So if say if we take the point (2, 3), if I put in 2 for x
420 Frank: Uhmhm.
421 Liz: 3 for y then that would make the equation true.
422 Frank: Well yes.
423 Liz: Now I don't know what c is at the moment,
424 Frank: yes
425 Liz: but if I know that 2 in there and 3 in there makes the equation true, then I can work out what c is.
426 Frank: Yes. It's 8 plus something equals 3.
427 Liz: Hmm.
428 Frank: ... 8, it would be 8 minus 5.
429 Liz: Yes.
430 Frank: Yes. So that's got to be -5. So it's got to be $y = 4x - 5$. (*writes $y = 4x - 5$*)

431 Liz: Uhmhm. Right, now the job that you've been given

432 Frank: Uhmhm.

433 Liz: is to find the equation of a line

434 Frank: yes

435 Liz: which doesn't have them specified as numbers.

436 Frank: Yes. So that, $q = mp - c$. (*writes $q = mq - c$*)

437 Liz: No do you mean p there?

438 Frank: I do mean p , not q . .. p . I can't write either.

439 Liz: What did you use that equation for when you were doing the other one?

440 Frank: That?

441 Liz: Hmm.

442 Frank: I used .. p is x because it's the x coordinate

443 Liz: Uhmhm.

444 Frank: q is y , because it's the y coordinate. (*writes $q = mp + c$*)

445 Liz: Hmm.

446 Frank: m is the gradient and c is the constant. And because I didn't know the constant but because I knew the other ones

447 Liz: right

448 Frank: I knew that $mp + c$ had to equal q , so I could just work out what c was.

449 Liz: Right. Well the same is true for this case.

450 Frank: Yes. So it would be $q = mp$. (*writes $q = (mp)$*)..... a bit of a shot in the dark ... 5 is what the two coordinates were when added together.

451 Liz: Hmm, bit of a coincidence.

452 Frank: Yes it is. I was just trying to work out how I could use this to find out what c is cos ..

453 Liz: Uhmhm. Let me write something down up here because when you were doing this you missed out the writing really. You just looked at this and put the figures in in your head. Umm, what

you were actually saying was here's 8 (*writes 8 beneath mx in the equation $y = mx + c$*) umm and this is 3 (*writes 3 = beneath $y =$*).

454 Frank: Uhmhm.

455 Liz: And I had to find out c (*writes $+ c$ beneath $+ c$*) umm and you did it in your head and you worked out that c had to be -5

456 Frank: Uhmhm

457 Liz: and then you just put it in straight away.

458 Frank: Yes.

459 Liz: Hmm. Now what you could have done here was do some rearrangement on paper rather than in your head which would say c must be $3 - 8$. (*writes $\Rightarrow c = 3 - 8$*)

460 Frank: Hmm. Yes.

461 Liz: You see where I got that, which gives you the -5 (*writes $= -5$*). Now you could, you could do that in your head because it's numbers.

462 Frank: Yes.

463 Liz: Umm but if you were dealing with letters

464 Frank: yes

465 Liz: it might be helpful to actually do that stage on paper.

466 Frank: q take away mp , (*writes $c = q - (mp)$*) this one - it's knowing q to take away from it

467 Liz: Hmm. Well there's a lot of not knowing in this question.

468 Frank: Yes, there is.

469 Liz: Because you don't know what m is, you don't know what p is, you don't know what q is.

470 Frank: Yes, or c .

471 Liz: Or c . Umm but they, can you see that not knowing what m and p and q are is different from not knowing what c is?

472 Frank: Yes. .. Because c is, q and p could be anything.

473 Liz: Hmm.

474 Frank: But c is the constant which puts them all together ...

475 Liz: Now although you don't know what q and the p are,

- 476 Frank: Uhmhm
- 477 Liz: you're going to carry on not knowing what they are because the question's never going to tell you.
- 478 Frank: Yes. (inaudible) you never get to know what c is.
- 479 Liz: Right. Yes. Tell me about that equation that you've just written down at the end there (*the equation reads $q = (mp) + (q - (mp))$*)
- 480 Frank: $q = mp +$ erh open brackets $q - mp$.
- 481 Liz: Yes.
- 482 Frank: Closed brackets. It's because c is if I was working it out with numbers
- 483 Liz: Uhmhm
- 484 Frank: c would be $q - mp$.
- 485 Liz: Right.
- 486 Frank: But as I don't know what $q - mp$ is, that's the closest I'm going to get to what c is
- 487 Liz: Okay. So you've worked out something for c .
- 488 Frank: Yes.
- 489 Liz: And we have to be satisfied with that.
- 490 Frank: Yes, yes.
- 491 Liz: But what are we going to do with this 'something for c '?
- 492 Frank: Erh I don't know.
- 493 Liz: Have a look back at the one you did with numbers and see what you did with it.
- 494 Frank: Once I'd worked it out, I put it back in.
- 495 Liz: Hmm. Put it back in what?
- 496 Frank: Into the equation there. (*points to $y = mx + c$*)
- 497 Liz: Yes. This is the equation of a line isn't it? (*pointing to $y = 4x - 5$*)
- 498 Frank: Yes.
- 499 Liz: Is this the equation of a line? (*pointing to $q = (mp) + (q - (mp))$*)

- 500 Frank: Uhmhm. Erh if it had the x and y in
- 501 Liz: h m m
- 502 Frank: you've got to change all these back to x and y yes? It would be general only (*writes $y = (mx) + (y - mx)$*).
- 503 Liz: Are you happy with that?
- 504 Frank: Hmm. Yes because like with the last one we did using the factorising
- 505 Liz: you couldn't go any further?
- 506 Frank: Yes, couldn't go much further and once you'd worked, I'd worked out the umm general I could see how everything else was being replaced and that it was the general form for the straight line.
- 507 Liz: Okay. Yes, it's, it's not quite as good as you can get actually.
- 508 Frank: No.
- 509 Liz: It's pretty close. But you'd be better to leave the last bit as $q - mp$
- 510 Frank: oh right
- 511 Liz: because otherwise this equation that you're giving me at the finish doesn't have p 's and q 's in it
- 512 Frank: right
- 513 Liz: and since I've told you that it goes through that point, I think it ought to have,
- 514 Frank: yes
- 515 Liz: umm but you're right in that you needed to leave some x s and y s in but not too many. (*Frank writes $y = (mx) + (q - (mp))$*) Okay. Right. Thank you ever so much for doing that Frank.

- 1 Liz: I, I want you to work through them basically, umm, but I'd like you to tell me as much as possible about what you're doing, umm, and also I'm going to ask you a few umm questions about what the question's about, that sort of thing. So, have a look at them. If there's any one you'd like to start with, you can have a choice of the first one.
- 2 Kevin: Hmm.
- 3 Liz: Erh would you rather leave now? (*laughter*)
- 4 Kevin: Probably. Er Right, I think we'll start with the first one because I can do most, some of that. (*the first question reads 'Factorise $x^3 - 1$, $x^3 - 8$ and $x^3 - a^3$ '*)
- 5 Liz: Okay, good.
- 6 Kevin: All right then. Umm, so because it's x^3 we have three brackets .. each with x in. And it's -1 , so really it will, erh, no hang on. I thought I knew how to do it.
- 7 Liz: Well, it's, it's quite a good start. Umm certainly
- 8 Kevin: I mean what it would be, it would be like, erh a plus, a plus and a minus, or three minuses
- 9 Liz: yes
- 10 Kevin: but then if you add them together you get 3, so it would be $x^3 + -1$, well, $-3x + -1$ or -1 , so that erh doesn't quite work.
- 11 Liz: Well, cubics are just a little bit more complicated than you're making it, because in a cubic generally you'd have x^3 , $2x^3$ or something.
- 12 Kevin: Ah hmm.
- 13 Liz: Then you'd have so many x^2 's
- 14 Kevin: ah hmm
- 15 Liz: then you'd have so many x s
- 16 Kevin: yes
- 17 Liz: and then you get your number term, so you've got four terms to worry about.
- 18 Kevin: So, the somehow, I've got to get $0x$ haven't I
- 19 Liz: yes

- 20 Kevin: and $0x^2$.
- 21 Liz: Hmm.
- 22 Kevin: Now so if you make one of these a minus. (*referring to the xs he has put at the beginning of each of the three brackets*)
- 23 Liz: Well you can't do that because you've got to have x^3 to start.
- 24 Kevin: Oh yes.
- 25 Liz: I'm just going to see if they'll mind if we shut this door. (*I go off to close the classroom door to shut out the extraneous noise*)
..... The way you're doing it
- 26 Kevin: well you could have two of them a minus couldn't you, because $-x$ by $-x$
- 27 Liz: yes
- 28 Kevin: is x^2 by another x is
- 29 Liz: ah hmm
- 30 Kevin: x^3 and that will give you $-x$, no it will give you $-2x$, well $-2x^2$.
- 31 Liz: Hmm. But the problem is that actually this one doesn't factorise into three brackets like that.
- 32 Kevin: Ah, so it factorises x into 2 (*he writes $x(\quad)(\quad)$*)
- 33 Liz: No.
- 34 Kevin: Like that, no?
- 35 Liz: No it can't do that because if it did then there wouldn't a number term in the end.
- 36 Kevin: Oh yes.
- 37 Liz: What, it would actually factorise as a bracket and the, and one other bracket and the other bracket is a quadratic,
- 38 Kevin: ah hmm
- 39 Liz: which won't factorise into two brackets.
- 40 Kevin: Right.
- 41 Liz: Right, so it's an irreducible quadratic. It's one that you can't do this to.

- 42 Kevin: Right, so we would have, well x^2 obviously, x and x^2 , umm -1 it wouldn't be an x because x times no x is zero. *(he is trying to decide on the x term in the quadratic)*
- 43 Liz: ... Yes, but, umm .. you're, you're thinking about that giving you the x^2 term, are you?
- 44 Kevin: Yes.
- 45 Liz: You've also got these two. *(I point out x^2 and the number term in the linear bracket)*
- 46 Kevin: Oh yes. So you want minus, $-x$ there ... It all depends on what that,
- 47 Liz: What are you going to have here? *(at the end of the linear term)*
- 48 Kevin: I was thinking of 1.
- 49 Liz: Hmm.
- 50 Kevin: -1 so that would be plus x so that would give you $1x^2$ and then that will give you $-1x^2$. Umm and then $+1$. No, that won't work, because that gives you zero.
- 51 Liz: What gives you zero?
- 52 Kevin: -1 and 1, no it's -1 times 1, ain't it
- 53 Liz: yes
- 54 Kevin: that's right. So it's, so it is right. Erh, yes. That's right. I think.
- 55 Liz: What can you do to check it?
- 56 Kevin: Multiply it out I suppose can't you
- 57 Liz: h m m
- 58 Kevin: so it would be $x^3 - 1x^2$ and x^2 and yes plus that -1 $x^3 - 1$, yes.
- 59 Liz: Can you just show me how you got those four terms.
- 60 Kevin: Er x^3 that and that. *(he shows me that x^3 is the product of x and x^2)*
- 61 Liz: Ah hmm.
- 62 Kevin: Er $1x^2$ *(product of x and x)* that and that will give you the other x^2 *(product of -1 and x^2)*
- 63 Liz: Ah hmm.
- 64 Kevin: And then that and that one. *(pointing to $+1$ and -1)*

- 65 Liz: Right, you've missed a couple out.
- 66 Kevin: Oh
- 67 Liz: You've done, you've taken the x^2 , you've done x times x^2 and you've done -1 times x^2 .
- 68 Kevin: Ah hmm.
- 69 Liz: Then the x
- 70 Kevin: Oh yes, that $-1x$ and x , so it's $-1x$ and $1x$.
- 71 Liz: Yes.
- 72 Kevin: So it still makes zero.
- 73 Liz: Okay.
- 74 Kevin: I would have passed in an exam wouldn't I, just because I didn't write it down. I got the right answer. *(laughter)*
- 75 Liz: Umm, can you remember about the factor theorem?
- 77 Kevin: No.
- 78 Liz: Hmm, well, it's a way of working out what the linear factor, this, the short bracket is, by doing a bit of guess work and trying it. What it says is that if umm 1 comes to 0 when you put it in here, if we put in $x = 1$ you get 0
- 79 Kevin: ah hmm
- 80 Liz: if you put $1^3 - 1$, umm then that means that $x - 1$ is a factor.....
- 81 Kevin: Yeah
- 82 Liz: Umm because if you put $x - 1$ into this, sorry if you put $x = 1$ into this
- 83 Kevin: ah hmm
- 84 Liz: then you get nothing.
- 85 Kevin: Oh yes.
- 86 Liz: Because that's nothing. .. Umm so if you get umm nothing when you put it into here, then there must be a factor like this in there.
- 87 Kevin: That's got nothing, the answer will be nothing.
- 88 Liz: Yes, yes.
- 89 Kevin: Right.

- 90 Liz: .. So that would have enabled you to work out that term, and then you know how to work out the other term, once you've made a guess at that one.
- 91 Kevin: Sort of.
- 92 Liz: No, that method's fine, it's the quickest way to do it. Other people sometimes use long division but that's so complicated
- 93 Kevin: yes I've messed that up on lots of them (*laughter*).
- 94 Liz: So, have a look at the next one.
- 95 Kevin: Right. $x^3 - 8$. Well using the same thing again, erm, if x is equal to 8 it will be 0 won't it.
- 96 Liz: $x^3 - 8$
- 97 Kevin: No, no, because of the cube, so it will be 2. So it would be 2, so the $x - 2$ which would be that one. And then x and into $-2x^2$ to equal 2, $2x$ and you $-4x$ so that would be $2x^2 - 4x$ $4x + 4$ times yes, that's right. That's right. (*he has written $x^2 + 2x + 4$*)
- 98 Liz: Okay. Umm have a look at the comparison between your first answer and your second answer. And then see what you can make of the third one.
- 99 Kevin: Umm, I don't know. I can't work that one out, I must I think it would be x plus something, or x minus something whichever. x would get minus That's not going to work. I don't know how you get the a^3 at the end. I mean because you like add them together
- 100 Liz: no
- 101 Kevin: unless that's a^2 .
- 102 Liz: Yes, it's not add is it. It's multiply.
- 103 Kevin: Oh yes, so it is. Yes, obviously, that one's the add one. So that gives you the a^3, ax^2 .. 0 and $-ax$ begin, no a^2x well I think I'm confusing myself. It's sort of right, but it's not quite (*he has written $(x + a)(x^2 - ax - a^2)$*)
- 104 Liz: Hmm.
- 105 Kevin: Because that gives you $-a^2x$.
- 106 Liz: Look at your first bracket. You had $x - 1$, $x - 2$, $x + a$.

- 107 Kevin: Well I suppose if that was a minus you'd have to make that a plus. and that one, (*he changes $-ax - a^2$ to $+ax + a^2$*) ... but that's still not quite right. It gives you a^3 , it gets, gets rid of the ax . .. And the ax^2 No it doesn't, it gets rid of the ax^2 but not the ax .
- 108 Liz: Where does the ax
- (*break in tape*)
- 109 Kevin: ax a , a^2 ah there's a^2x
- 110 Liz: hmm
- 111 Kevin: ah yes. So there's a^2x again. Like that. That's right isn't it?
- 112 Liz: Yes.
- 113 Kevin: I'll just put me signs round the right way.
- 114 Liz: Okay. Now have a look at your three answers.
- 115 Kevin: Oh they're all the, the sort of same aren't they, because the last one's a^2 which would be this, is this squared, and it's this squared again and that's this squared again.
- 116 Liz: Uhmhm.
- 117 Kevin: So that's a , what is it a factor, a factor of this. No? Something like that.
- 118 Liz: What's a factor of what?
- 119 Kevin: This one is a factor of this. (*Indicates that $x - 2$ is a factor of $x^2 + 2x + 4$*)
- 120 Liz: What makes you say that?
- 121 Kevin: Or something, something like that. Umm because it's, it just seems to go into it and well
- (*We are interrupted by another pupil*)
- 122 Kevin: .. just an idea.
- 123 Liz: I'm not sure what you mean by that's a factor of that.
- 124 Kevin: It means it will go into it, like you could divide by that and end up with nothing.
- 125 Liz: Right.
- 126 Kevin: Well no remainder

- 127 Liz: No, you couldn't.
- 128 Kevin: No?
- 129 Liz: Umm none of these three have any factors (*pointing to $x^2 + x + 1$, $x^2 + 2x + 4$ and $x^2 + ax + a^2$*). They don't have, you can't put them into two brackets.
- 130 Kevin: Oh right.
- 131 Liz: They are not umm, they're irreducible. They can't go any further than they've got. Umm .. but when you say they're all the same
- 132 Kevin: yes, $x - x - x^2$ + something x + something
- 133 Liz: yes. So in fact, if you'd worked the third one out first
- 134 Kevin: ah hmm you could have done that a lot easier. Well I could have done a lot easier instead of erh.
- 135 Liz: Could you factorise um, $x^3 - 27$?
- 136 Kevin: .. Er probably. put 3 in, $x - 3$, $x^2 + \dots$ and $x - 3$, that's a 9 that's x^2 and this is $3x$.
- 137 Liz: Yes, how do you know it's got to be $3x$ in the middle?
- 138 Kevin: Umm well because I looked at this again, the ones before where it's a , it's ax where it's 2, it's $2x$, where it's 1 it's $1x$
- 139 Liz: Right, so you're following the pattern?
- 140 Kevin: Ah hmm.
- 141 Liz: Fine, good, that's great. I'm glad you don't do long division, I hate it.
- 142 Kevin: I can't do it that's why. I keep messing it up, I try.
- 143 Liz: Umm, all of these questions are hard.
- 144 Kevin: Ah hmm.
- 145 Liz: Umm, I mean, deliberately so. Have a go at number 6. (*Number 6 reads 'Find the equation of the tangent to the curve $y = x^2 + 1$ which passes through the origin'*)
- 146 Kevin: Hmm. I'll draw it first. It may not be any help but
- 147 Liz: it gives you something to do while you're thinking about it.
- 148 Kevin: Yes. So it will be $y = x^2 + 1$...

- 149 Liz: I'm just going to shut this door (*there is a lot of noise from outside the room*)
- 150 Kevin: No hang on. x^2 is (*he has drawn a sketch of $y = x^2 + 1$*) So that would be, .. it passes through the origin, that's the line that passes through the origin, the tangent line?
- 151 Liz: Yes.
- 152 Kevin: Hang on I'm trying that line there. No that's wrong. .. Umm well $y = x$ goes through the origin. but it's not the tangent to the curve.
- 153 Liz: Well no not necessarily no.
- 154 Kevin: No, because it doesn't, doesn't touch it, does it?
- 155 Liz: Hmm.
- 156 Kevin: Because it's, a little bit, a little bit off because it would have to go through (1, 1) and it doesn't.
- 157 Liz: Right.
- 158 Kevin: .. Umm, so it's gotta be something similar. Trying to think how you make it steeper. Umm because $x + 1$ moves it umm moves it up one, so
- 159 Liz: hmm, that's right
- 160 Kevin: we don't want like, or it won't go through the origin.
- 161 Liz: Hmm.
- 162 Kevin: Okay
- 163 Liz: Can you remember about $y = mx + c$?
- 164 Kevin: I was just thinking that.
- 165 Liz: Yes. Let me remind you what that means. Then it will probably help you.
- 166 Kevin: The gradient m , so that would be $2x$, something like $2x$ wouldn't it?
- 167 Liz: Yes that's right. That tells you where it goes through the y -axis (*pointing to c in $y = mx + c$*)
- 168 Kevin: h m m
- 169 Liz: and this one's the gradient. (*pointing to m*)

- 170 Kevin: Yes so it's something like $2x$ would probably be, would probably be I think, hang on if you put a 1 in, you get when it goes through (1, 2), so $2x$, $y = 2x$ would be the tangent, the erh equation of a tangent.
- 171 Liz: Is that the only one?
- 172 Kevin: Umm no because there could be one going that way couldn't there? (*shows a tangent with negative gradient*)
- 173 Liz: Ah hmm.
- 174 Kevin: ... So it would be , that's, that's the normal to the tangent so that would be, you, it would be one over two x .
- 175 Liz: Yes, it's not at right angles to it actually.
- 176 Kevin: No?
- 177 Liz: No.
- 178 Kevin: Ahh.
- 179 Liz: I mean it looks a bit like it on your diagram, but if I can draw this tangent here, the other one's actually a bit steeper than you've drawn it as well. That's not a right angle. It's less than a right angle.
- 180 Kevin: Oh right.What, would it just be $-2x$.
- 181 Liz: Yes.
- 182 Kevin: Makes it symmetrical
- 183 Liz: Okay so umm what makes you think that's the tangent?
- 184 Kevin: .. Umm well it's a line that passes through the origin
- 185 Liz: yes
- 186 Kevin: and it's got to touch the curve somewhere. ... oh no, it's only got to just touch the curve, it hasn't got to cross it.
- 187 Liz: Yes, that's right.
- 188 Kevin: So it's like at the, at the lowest, well not the lowest point, erh, as near as you can get to it
- 189 Liz: yes
- 190 Kevin: without crossing it.
- 191 Liz: So you know that that line goes through the point (1, 2)

- 192 Kevin: ah hmm
- 193 Liz: and you know that the curve goes through the point (1, 2).
- 194 Kevin: Yes.
- 195 Liz: But you're not absolutely sure that it touches it there. It could just cut it there, couldn't it?
- 196 Kevin: Yes. It could do. So you, should try like something like that's 2, (2, 5) (*pointing to the equation $y = x^2 + 1$*), and then if you put 2, (2, 4) (*pointing to the equation $y = 2x$*) so it's gone and moved away again
- 197 Liz: hmm
- 198 Kevin: the line's moved away from the, the erh parabola.
- 199 Liz: .. Right, yes it has by the time you've got to $x = 2$.
- 200 Kevin: Ah hmm.
- 201 Liz: Yes.
- 202 Kevin: So it's sort of a good indication that it's, it's not brilliant but it's
- 203 Liz: yes. You're right actually. That is the tangent but you couldn't be absolutely sure that it was on the basis of the reasoning that you've done.
- 204 Kevin: Oh no.
- 205 Liz: What you've missed is that the curve and the tangent actually have the same gradient there, at that point. That's how you know that it's a tangent. That it touches it and then skates off again. The gradient that the curve has at that point
- 206 Kevin: yes of course
- 207 Liz: has got to be the same as the gradient
- 208 Kevin: $2x$
- 209 Liz: of the line.
- 210 Kevin: Ah hmm.
- 211 Liz: So, sorry, of course it is what?
- 212 Kevin: Well when you differentiate $x^2 + 1$ it's $2x$.
- 213 Liz: Ah hmm. So what's -
- 214 Kevin: And if you put in 1, that gives you 2, and if you put 1 in at this bit it gives you 2 as well.

- 215 Liz: Right. Okay. Umm
- 216 Kevin: So that, that does prove it.
- 217 Liz: Yes. Right. Same question except that now the curve is $y = x^2 + 2$.
- 218 Kevin: .. Right. That will be the same, that will be the same again wouldn't it? .. Ah no it wouldn't. .. Does it still, it still passes through the origin?
- 219 Liz: Yes.
- 220 Kevin: So I think it would probably be $3x$. No it wouldn't. No it should still be $2x$
- 221 Liz: h m m
- 222 Kevin: because it doesn't change when you differentiate it does it? $x^2 + 2$ is the same gradient as $x^2 + 1$.
- 223 Liz: Ah hmm.
- 224 Kevin: So the, the gradient of this line has got to be the same, $2x$ for it to be the tangent. Yes
- 225 Liz: Well, just now when we were talking about the gradients, you said the gradient of that curve is $2x$.
- 226 Kevin: Yes, well I'm, I'm
- 227 Liz: So at this point
- 228 Kevin: yes, they'll meet in a different point further up this, this umm erh line.
- 229 Liz: Hmm.
- 230 Kevin: But it will still be the same, the same gradient at that one point.
- 231 Liz: .. So it will be a different x coordinate, the point where they touch.
- 232 Kevin: Ah hmm. I think that's all there'll be different.
- 233 Liz: Well in that case, won't the gradient of the curve have changed?
- 234 Kevin: Oh it will have moved, it moves up one doesn't it?
- 235 Liz: Ah hmm.
- 236 Kevin: But that's all it does. It stays the same, the same shape. So the gradient and the, on the, at this point on $x^2 + 1$ is the same as this point on $x^2 + 2$.
- 237 Liz: Ah hmm.

- 238 Kevin: So the gradient would be the same, it would just be in a different, up one higher.
- 239 Liz: Yes, in which case, the line $y = 2x$ won't touch it at
- 240 Kevin: no it will need to be
- 241 Liz: the place where $x = 1$
- 242 Kevin: It would need to be $2x^2 + 1$ to touch it at the same place, wouldn't it?
- 243 Liz: Yes.
- 244 Kevin: But then it doesn't go through the origin.
- 245 Liz: That's right.
- 246 Kevin: So that defeats the object.
- 247 Liz: Hmm. (*I begin programming a graphic calculator*).....
I've drawn $y = x^2 + 2$ and $y = 2x$.
- 248 Kevin: Ah hmm.
- 249 Liz: If we put in $y = x^2 + 1$ as well.
- 250 Kevin: So it would need to be steeper wouldn't it?
- 251 Liz: Hmm.
- 252 Kevin: Well no because it would still touch it eventually wouldn't it?
- 253 Liz: Umm ah let's have a look. (*I zoom out on the graph*)
- 254 Kevin: Ah no it doesn't. .. Hmm.
- 255 Liz: So as you say, it would have to be steeper wouldn't it?
- 256 Kevin: Hmm. .. But then that messes up the, oh I suppose there must be, must be a bit on this which has got a gradient of $3x$ and .. somewhere.
- 257 Liz: A gradient of what?
- 258 Kevin: I mean, I was thinking that you put it up, I mean it's not going to be much steeper so $3x$ would probably
- 259 Liz: h m m
- 260 Kevin: be enough to make it touch. And there should be somewhere around here where the, oh no it's not though, is it? Because I think

261 Liz: Somewhere round there where the gradient - ?

262 Kevin: is $3x$ but there's not.

263 Liz: Somewhere round there where the gradient is 3 is what you want.

264 Kevin: Yes, that's right. ... Hmm, so bit of a dodgy number!

265 Liz: u m m

266 Kevin: for x that is

267 Liz: well let's try $y = 3x$.

268 Kevin: Okay. (*we add it to the graph screen on the calculator*)
That crosses.

269 Liz: Right. Yes I think it does, so it's going to be something between 2
and 3 isn't it?

270 Kevin: Yes.

271 Liz: Umm guessing might not get us very far very quickly.

272 Kevin: Hmm no.

273 Liz: Because I mean it could be something really awkward,

274 Kevin: h m m

275 Liz: like 2 and $11/12$ ths. Umm, so I suggest this. Let's suppose that we
know what the equation of the line is. We know it's $y = (2 \text{ and a bit}) \text{ times } x$.

276 Kevin: Ah hmm.

277 Liz: So let's put in a letter for the 2 and a bit.

278 Kevin: Right.

279 Liz: Like a ,

280 Kevin: ah hmm

281 Liz: so we know the tangent is $y = ax$.

282 Kevin: Right.

283 Liz: And let's see if we can work out what a has to be.

284 Kevin: Oh I see, by so, well the gradient of the curve is still going to be
 $2x$ ain't it?

285 Liz: Ah hmm.

- 286 Kevin: So the gradient of the curve, $2x$. (*writes gradient $y = 2x$*)
- 287 Liz: ... When you say that the curve has a gradient of $2x$, what does that mean?
- 288 Kevin: Umm, if you put, well if you look at where x is 1, the gradient's 2, where x is 2, the gradient's 4, things like that
- 289 Liz: Okay. And we're interested in the point where the gradient is a .
- 290 Kevin: Ah hmm. So it's hmm the gradient's a , so we want 2 times half of a don't we, the gradient
- 291 Liz: hmm
- 292 Kevin: x will be half a . So it still doesn't help.
- 293 Liz: Well it means that the point that we're looking for, where the line and the curve touch each other
- 294 Kevin: hmm
- 295 Liz: is where $x = \frac{1}{2}a$.
- 296 Kevin: Yes, okay. (*writes $2 \frac{1}{2}a$* .)
- 297 Liz: What does that mean, 2 times $\frac{1}{2}a$?
- 298 Kevin: Yes.
- 299 Liz: .. Why have you written that down, what does it mean?
- 300 Kevin: Well the, that's, well the gradient is a ain't it?
- 301 Liz: Right.
- 302 Kevin: So by 2 times
- 303 Liz: so that's the gradient?
- 304 Kevin: Yes.
- 305 Liz: Okay.
- 306 Kevin: It may not be any help but, I thought I'd write it down
- 307 Liz: Right. So this point on the tangent (*writes $(\frac{a}{2}, \quad)$*) is umm x coordinate is $\frac{1}{2}a$
- 308 Kevin: ah hmm

- 309 Liz: what's the y coordinate?
- 310 Kevin: Umm, it will be $a + 1$ won't it? no, $\frac{a}{4} + 1$ wouldn't it?
- 311 Liz: $\frac{a^2}{4} + 1$, yes.
- 312 Kevin: Well no, will it be $\frac{1}{2}a^2$ because that's x isn't it?
- 313 Liz: It's $\frac{1}{2}a$ all squared, yes.
- 314 Kevin: Ah hmm. ..+1.
- 315 Liz: Yes. Can you write that down so I know what you mean. (*Kevin writes $(\frac{a}{2})^2 + 1$ in the space I have left*) Okay. And how do you know that's the y coordinate?
- 316 Kevin: Through by x , oh hang on that's wrong. We're doing 2 aren't we? we're doing $x +$, $x^2 + 2$ aren't we?
- 317 Liz: Oh yes, that's right.
- 318, Kevin: Yes. So 2, it'd be + 2. (*he replaces the 1 with a 2*)
- 319 Liz: Okay. So you know that's the y coordinate because it's on the curve.
- 320 Kevin: Ah hmm.
- 321 Liz: You're using the equation of the curve. Also you know that that point's on the tangent.
- 322 Kevin: .. Yes, so it's 2 and I dunno, 2 and whatever $(\frac{1}{2}a)^2$ is
- 323 Liz: go on
- 324 Kevin: it's, so, so the point where it touches is 2 and whatever $(\frac{1}{2}a)^2$ is. 2 and a little bit.
- 325 Liz: .. Right. And that, the coordinates of that point also satisfy the equation of the tangent.
- 326 Kevin: Ah hmm, so it's going, so whatever goes in there has got to be equal to this one yeah?
- 327 Liz: Hmm. The coordinates that we've worked out so far
- 328 Kevin: h m m

- 329 Liz: we said that that point the x coordinate is $\frac{1}{2}a$.
- 330 Kevin: Ah hmm.
- 331 Liz: It had to be a $\frac{1}{2}a$ in order for the gradient to be a .
- 332 Kevin: Yes.
- 333 Liz: Yes. And the y coordinate is that (*indicating* $(\frac{a}{2})^2 + 2$)
- 334 Kevin: ah hmm
- 335 Liz: because of it lying on the curve. But also this point lies on that tangent, so it fits that equation too. (*pointing to the equation* $y = ax$)
- 336 Kevin: Yes.
- 337 Liz: So you've got to say the y coordinate is a times the x coordinate.
- 338 Kevin: Ah hmm.
- 339 Liz: Which means the y coordinate is a times the x coordinate. (*writes* $(\frac{a}{2})^2 + 2 = a \cdot \frac{a}{2}$)
- 340 Kevin: Right, yes.
- 341 Liz: Yes. Which now gives you an equation for a So you've got an equation there that's only got a s in it
- 342 Kevin: ah hmm
- 343 Liz: so if we could work it out and solve it, we'd know what a was. Are you thinking about how to solve it?
- 344 Kevin: Ah hmm.
- 345 Liz: Um well I wouldn't worry too much about that (*laughter*). That's the easy part. I'll do that.
- 346 Kevin: I was thinking about timesing it all by 2.
- 347 Liz: Yes. I would, I would work this out first. (*pointing to* $(\frac{a}{2})^2$)
- 348 Kevin: Oh right.
- 349 Liz: That's $\frac{a^2}{4}$ Yes?

- 350 Kevin: Ah hmm.
- 351 Liz: Now you could times it all by 4.
- 352 Kevin: Oh right, so we get $a^2 + 8 = 2a^2$. Yes?
- 353 Liz: Ah hmm.
- 354 Kevin: Umm which way to do which, umm so if we take a^2 , leave that, no we don't want to do that.
- 355 Liz: I would.
- 356 Kevin: Well doesn't that, oh I see that leaves, that leaves a^2 doesn't it?
- 357 Liz: Ah hmm.
- 358 Kevin: $8 = a^2$ so $a = \dots$ what's the square root of 8?
- 359 Liz: hm, it's the square root of 8!
- 360 Kevin: I just can't see things like that. It takes me ages.
- 361 Liz: So now we've got a
- 362 Kevin: ah hmm
- 363 Liz: what does that mean?
- 364 Kevin: Erh we can find out the gradient. So it's $\sqrt{8x}$ the tangent gradient of the line, the equation.
- 365 Liz: Okay. Do you want to try it out? (*I offer him the graphic calculator*)
- 366 Kevin: How do you do roots on this?
- 367 Liz: Use the square root.
- 368 Kevin: Oh right. Just - first?
- 369 Liz: Yes.
- 370 Kevin: So, $\sqrt{8x}$.. (*the graph appears*) oh that's very strange.
- 371 Liz: Oh I think it's included the x after the square root. It's done the square root of $8x$. Instead of the square root of 8, times x .
- 372 Kevin: Ah, right. We'll try that again then. .. So it would be umm bracket, square root, eight, bracket x yes?
- 373 Liz: Yes.
- 374 Kevin: Mm, it sort of rubs alongside it.

- 375 Liz: Yes, I'm glad you can see that. I can't make it out with that screen
- 376 Kevin: I saw it - I saw it draw it
- 377 Liz: Right.
- 378 Kevin: It's just a bit, just a bit further over than $3x$.
- 379 Liz: Okay. That was quite a hard question.
- 380 Kevin: No! (*laughter*)
- 381 Liz: Umm if we go back to where I said 'let's make the curve $x^2 + 2$ '
- 382 Kevin: ah hmm
- 383 Liz: instead of $x^2 + 1$, could you tell me about the steps that you went through?
- 384 Kevin: .. Umm .. what to, to get where we got?
- 385 Liz: Yes.
- 386 Kevin: Oh right. We, we deducted that it wasn't $2x$ any more, the tangent and it, and $3x$ was too much.
- 387 Liz: Ah hmm.
- 388 Kevin: So it had to be in between 2 and $3x$. And we didn't know what so we called it a this thing
- 389 Liz: ah hmm
- 390 Kevin: Umm and we still know the gradient of the curve is $2x$. So for point, .. no hang on, erh, this point, this point on the curve .. would have to be $\frac{1}{2}a$, yes, that's right. This point on the curve at where the, where they touch the tangent and the curve touch would have to be $\frac{1}{2}a$, so the gradient at that point would be 2, 2 times $\frac{1}{2}a$ because it's on the curve, the gradient. So oh yes, we put 2, erh, yes $\frac{1}{2}a$ into the $y = x^2 + 2$ and we've come up with the two coordinates, $\frac{a}{2}$ and $(\frac{a}{2})^2 + 2$. Umm I'm trying to remember where we got this bit from. No. Oh yes, right. This is, we got the gradient of the, the curve is equal to the gradient of the tangent, which is a times x , which is, the x is now $\frac{a}{2}$, so then, we squared ax , erh squared $\frac{a}{2}$ and got $\frac{a^2}{4}$ and then added the two, which

is equal to $\frac{a^2}{2}$, which is a times $a, \frac{a}{2}$. I've got it now. I know where we are.

391 Liz: Hmm.

392 Kevin: And then erh re-arranged the formula

393 Liz: yes

394 Kevin: to er -

395 Liz: Yes, okay.

396 Kevin: Get, get $\sqrt{8}$

397 Liz: Right. Could you say anything about why this question is so hard? What makes it difficult.

398 Kevin: Umm, it being a dodgy number.

399 Liz: (laughter)

400 Kevin: Like 2 point whatever it was.

401 Liz: Umm, oh I see what you mean. Okay. Yes, it is a bit.

402 Kevin: Hmm, I mean if it was a round number it would be really simple.

403 Liz: Because?

404 Kevin: Well I don't know, I just find whole numbers a lot easier.

405 Liz: Umm tell me about the difference between the way you do the first one, when it was $x^2 + 1$, and the way you did the second one

406 Kevin: Umm well it's because the gradient - the, the two, two gradients were the same. .. No. Well I sort of, sort of guessed. But erh $y = x$ didn't fit because it's one more, so we had to make it steeper, and $2x$ is steeper and it just fitted.

407 Liz: Right.

408 Kevin: And it had the same, the same gradient and umm the same points.

409 Liz: So you did that first one by a sort of trial and error method.

410 Kevin: Ah hmm.

411 Liz: How would you describe the method that you used for the second one?

- 412 Kevin: Umm probably doing it properly, using deduction and erh, and instead of guessing putting an unknown value in and then working out what the unknown value is.
- 413 Liz: Right.
- 414 Kevin: Instead of, instead of guessing the unknown value, working it out.
- 415 Liz: Okay. I think you've worked hard enough.
- 416 Kevin: Oh thank you.
- 417 Liz: Thank you very much Kevin.

Lorne 1/7/94

- 1 Liz: Good, I think the batteries are working.
- 2 Lorne: Okay.
- 3 Liz: What's that one then? *(I am asking which question he has chosen to start with)*
- 4 Lorne: Question 2. *(The point (a, b) is equidistant from the x-axis and the point (1, 2). Find an equation linking a and b)*
- 5 Liz: Oh that, yes, okay.
- 6 Lorne: I
- 7 Liz: Start with a tough one! *(We had done a very similar question in the lesson the previous day)*
- 8 Lorne: Yes, absolutely. Right. Umm draw x and y axes. .. x axis, ... (1, 2) .. there. *(he draws a pair of coordinate axes and marks the point (1, 2))*
- 9 Liz: Uhmhm.
- 10 Lorne: Okay. Umm, equidistant from .. that's going to be ... *(he discovers that his diagram is too small to include the detail he requires)...*, oh I'll redo it, I'll redo it.
- 11 Liz: Right.
- 12 Lorne: (1, 2), erh, it's going to be here and it's going to be a parabola *(he draws a rough sketch of the parabola)*
- 13 Liz: Uhmhm.
- 14 Lorne: Okay and find an equation for a, b. So .. if that point there is (1, 1) that would be the turning point.
- 15 Liz: Uhmhm.
- 16 Lorne: And if I call this x here *(he marks a point on the parabola between the turning point and the y-axis)* will equal x. Put it on the positive side there, which makes the distance from y the same as the distance from that point there. *(he marks the distance from his point to (1, 2) and from his point to the x-axis)*
- 17 Liz: Uhmhm.
- 18 Lorne: That's y *(he marks the distance from his point to the x-axis as y)* so that makes it y which equals $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
(writes $y = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$)..... x_1 is 1, minus x, squared, plus

2 minus y , squared (*writes* $= \sqrt{(1-x)^2 + (2-y)^2}$ which is equal to $1 - 2x + x^2 - 2 - 4y + y^2$. (*writes* $y = \sqrt{1 - 2x + x^2 - 2 - 4y + y^2}$)

- 19 Liz: Do you want a minus sign there? (*pointing to the minus sign before the 2*)
- 20 Lorne: Plus.
- 21 Liz: It's plus, yes
- 22 Lorne: Or we could y which becomes $y^2 = 1 + 2x +$ (*writes as he speaks* $y^2 = 1 + 2x + x^2 + 4 - 4y + y^2$)
- 23 Liz: $-2x$
- 24 Lorne: yes, $x^2 + 4 - y \dots 4y + y^2$ which cancels out
- 25 Liz: Uhmhm
- 26 Lorne: which means I can put this $4y$ on the left hand side, which makes umm $1 - 4$ which is 3 , so this bit I can make $1 - 2x + x^2 + 4$, $+ \dots 4$ that's it, which then becomes $4y = -2 + x^2 + 3$ (*writes this*) and it can be changed round to
- 27 Liz: Is that $-2x$
- 28 Lorne: $-2x$, yes (*corrects* -2 to $-2x$) - makes it equal to
- 29 Liz: $+5$
- 30 Lorne: yes (*changes the 3 to a 5*), - which then equals, which is really the same as $x^2 + 2x + 5$ (*writes this*) I've just put it round, the right way round.
- 31 Liz: Uhmhm.
- 32 Lorne: Then $y = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{5}{4}$. (*writes this*) .. That's the answer to it (*writes* *ANS by this last equation*).
- 33 Liz: Okay. Umm .. what was this question about then?
- 34 Lorne: Umm the equal distance in, the equal distance between the one point and the other point - locus.
- 35 Liz: Right. You said this was like a question that we've done.
- 36 Lorne: Uhmhm.
- 37 Liz: What's erh similar and what's different about it?

- 38 Lorne: Umm ... well it's the same, what I'm doing here is working out the equation of a , of the actual line, but the question says find an equation linking well linking a and b .
- 39 Liz: Uhmhm.
- 40 Lorne: Which is the same thing isn't it? Or is it?
- 41 Liz: You tell me.
- 42 Lorne: Umm .., yes.
- 43 Liz: So you haven't strictly speaking answered their question, have you?
- 44 Lorne: No, not quite, just an equation.
- 45 Liz: So if I was being umm pedantic and saying can I have an answer to the question please
- 46 Lorne: then umm that's not the answer.
- 47 Liz: (*laughter*)
- 48 Lorne: It's, it's
- 49 Liz: It was dangerous
- 50 Lorne: yes
- 51 Liz: to write 'answer' next to something.
- 52 Lorne: Yes. It's similar to what we've done but what you've got to do is find an equation linking a and b , and umm linking a and b , I mean, I don't know actually umm .. an equation linking a and b . I, I don't, I don't quite understand what finding the equation linking a and b really means.
- 53 Liz: Uhmhm. Well what erh what part does a and b play in this question?
- 54 Lorne: It means it's any point on the parabola which is this same length between the point (1, 2) and the x -axis.
- 55 Liz: Right so it's any point on that parabola that you've sketched.
- 56 Lorne: Yes.
- 57 Liz: Umm when you wrote this equation down
- 58 Lorne: Uhmhm
- 59 Liz: you were referring to a point on the parabola.

- 60 Lorne: Yes.
- 61 Liz: What did you call it?
- 62 Lorne: .. I called it, ... what do you mean what did I call, I mean I
- 63 Liz: Well you were talking about this point here weren't you?
- 64 Lorne: Yes.
- 65 Liz: What are the coordinates of that point?
- 66 Lorne: The coordinates of that point is umm (a, b) , (y, x) umm (p, q) .
- 67 Liz: Yes, quite. (a, b) or (x, y) .
- 68 Lorne: Any yes.
- 69 Liz: What, you used x and y .
- 70 Lorne: Ah ha. So what I could do is put erh, y is b , so $b = \frac{1}{4}a^2$ could I?
- 71 Liz: b equals a $\frac{1}{4}$ what?
- 72 Lorne: $\frac{1}{4}a^2$ +. So,
- 73 Liz: Yes
- 74 Lorne: I'm not, what I'm answering is the equation linking x and y
- 75 Liz: yes
- 76 Lorne: instead of a and b . So it's $b = \frac{1}{2}a^2 - \frac{1}{4}a + \frac{5}{4}$. (*writes this*)
- 77 Liz: Okay.
- 78 Lorne: Which is the answer. (*writes ANS by his last equation*)
- 79 Liz: Hmm. Right. Umm okay so one of the most obvious differences between this question and the ones that we did earlier is that it has (a, b) in it.
- 80 Lorne: Hmm.
- 81 Liz: Anything else?
- 82 Lorne: Umm ... no it's not, . yes okay, I wouldn't, I wouldn't say not really it's more or less similar.
- 83 Liz: .. Which question in particular are you thinking of that it's similar to?

- 84 Lorne: Umm the one with the parabola.
- 85 Liz: So it's the fact that you end up with a parabola that's the similarity.
- 86 Lorne: It's, well it's got equal distance, so it's the same, it's more like question one, but using y instead of, using it in terms of y well going back and y and z , no x using the terms of y instead of x .
- 87 Liz: Right.
- 88 Lorne: So it's a 'u' shape instead of a 'c' shape.
- 89 Liz: Uhmhm.
- 90 Lorne: really
- 91 Liz: So we're talking about this question, the example we did together, and question one. They're all similar.
- 92 Lorne: Umm, more or less except for it's different terms and that the shape's different. I wouldn't I wouldn't actually say it refers to question 1. I would say it more or less refers to the question that we did together on the board, which was a past exam
- 93 Liz: yes
- 94 Lorne: paper. That one question. There was that was y and, and yes.
- 95 Liz: Right. So they both gave you parabolas.
- 96 Lorne: Uhmhm.
- 97 Liz: Umm, what was similar and different about the way in which the conditions were set up?
- 98 Lorne: Umm well I mean I could speak clearly
- 99 Liz: What do you mean?
- 100 Lorne: Umm when you mean conditions, it'll mean
- 101 Liz: I mean the fact that this one was equal distant
- 102 Lorne: ahh
- 103 Liz: from the x -axis and the point $(1, 2)$. That's what I mean by the conditions.
- 104 Lorne: I think the question before was umm you know, I don't think it was point $(1, 2)$, I think it was close.
- 105 Liz: I think it was $(3, 2)$.

- 106 Lorne: Yes. So much may be (3, 2). otherwise it's more or less the same.
- 107 Liz: Hmm.
- 108 Lorne: Because it's using the x -axis as well.
- 109 Liz: Right. And question one was using a line $x = -2$.
- 110 Lorne: Yes. Which meant it's a, be a vertical line instead of horizontal.
- 111 Liz: Okay. Umm so from your experience of these questions, could you give me a general description of the conditions, you know what I mean by that now,
- 112 Lorne: h m m
- 113 Liz: which give rise to a parabola.
- 114 Lorne: .. Umm it's what a u-shaped parabola or would it be or c-shaped?
- 115 Liz: Either.
- 116 Lorne: Umm, it's got to be equal distance and it's got to be, yes?
- 117 Liz: Equal distance
- 118 Lorne: And but it's, it's got to from, from a point and a line that's level with it's parallel to either the x -axis or y -axis. Not
- 119 Liz: right
- 120 Lorne: a diagonal line or another curve.
- 121 Liz: Okay. What do you think would happen if you, if the conditions were equidistant from this point and from, let's say the line $x = y$.
- 122 Lorne: Umm, I don't know. What would it be like? It would be the same, it will be the same because but it won't be, it will be like a 'u' in the, the x -axis will be the y -axis.
- 123 Liz: Uhmhm.
- 124 Lorne: It will be slightly turned over .. otherwise it would be the same.
- 125 Liz: Yes, that's right. So in fact if the point is equidistant from any point and any line, any straight line
- 126 Lorne: any straight line yes,
- 127 Liz: it would be a parabola. But umm, the two cases that you've looked at are when the parabola's this way (*gesticulating a parabola with directrix parallel to the x -axis*)

- 128 Lorne: yes
- 129 Liz: or when the parabola's that way (*gesticulating a parabola with directrix parallel to the y-axis*). And as you say, if, if the line was at an angle,
- 130 Lorne: ah ha
- 131 Liz: the parabola would be at a funny angle as well.
- 132 Lorne: Yes.
- 133 Liz: Okay. So that's enough on that one. Umm, was there another one that you, (*Lorne turns to question 3 which reads 'Sketch $y = x(x - a)$ '*) okay.
- 134 Lorne: Umm $y = x(x - a)$. Umm, well, first of all, expand it – is that the correct term for it?
- 135 Liz: Yes.
- 136 Lorne: x^2 and umm – xa . $y =$, is that right, is that right? Umm that would probably be, I don't know I probably need a table or, perhaps. .. yes may as well. No. Umm
- 137 Liz: Think of some curve sketching techniques.
- 138 Lorne: Well it's going to be a .. parabola.
- 139 Liz: Uhmhm.
- 140 Lorne: It's going to, oh umm, I need my graphical calculator. I wouldn't, wouldn't do this, I wouldn't actually do this if I didn't have my graphics calculator. It's such an invaluable type of thing. I definitely need it.
- 141 Liz: Have you got it with you?
- 142 Lorne: Yes, I have.
- 143 Liz: Okay.
- 144 Lorne: (*he gets out his calculator*) I mean, what I'll probably do is, is a trial and error sort of phase but it won't be like an error. What I'll probably do is $a = 1$ and then 2, 3, so if this one's 1, then it's umm graph $x^2 - x$... right .. goes slightly, goes through the origin.
- 145 Liz: You've still got that on dot instead of connected, haven't you? (*I am talking about the graphing mode on his calculator*)
- 146 Lorne: Hmm.

- 147 Liz: It doesn't matter.
- 148 Lorne: Yes, umm, I don't know how to change it now. Umm connected 5. Okay. $x^2 - x$ which goes through origin and (0, 1). $a = 2$ slightly move it down and go to 2 I think. umm, $2x$ yes it goes through 2 sorry. I guess 3 ... will go through 3, origin and 3, (0, 3). So whatever a is it goes through, which means that .. if I was to generalise it, I'd draw a parabola which crosses at the origin and climbs back up at point (0, a).
- 149 Liz: Okay. Umm. Two things.
- 150 Lorne: Uhmhm.
- 151 Liz: How could you have got that from the equation without using your calculator?
- 152 Lorne: By umm, by, by actually working it out properly before I did it (*laughs*).
- 153 Liz: Yes, I mean, what, what would you have to work out? How would you do it?
- 154 Lorne: Umm well what I could do is draw a table and do what I did with a as 1, 2, 3. But, yes,
- 155 Liz: What about where, where it crosses the axes? That's the thing you immediately noticed on the calculator.
- 156 Lorne: Uhmhm.
- 157 Liz: Umm, do you know how to work that out from the equation?
- 158 Lorne: Well if it had a third term something like $x^2 - xa + 2$, you would know it was going to cross the y -axis at 2
- 159 Liz: Uhmhm
- 160 Lorne: and umm
- 161 Liz: and since it doesn't
- 162 Lorne: Sorry?
- 163 Liz: Since it doesn't.
- 164 Lorne: Since it doesn't it hasn't got a term it goes through zero doesn't it.
- 165 Liz: Right. Yes, it goes through the origin.
- 166 Lorne: So it doesn't have to say anything. Umm

- 167 Liz: What about where it crosses the x -axis?
- 168 Lorne: Umm I don't know. I mean I, I mean, I don't know, I guess that it crosses x at a .
- 169 Liz: Hmm, yes.
- 170 Lorne: It is a
- 171 Liz: Well it crosses the x -axis wherever $y = 0$.
- 172 Lorne: Yes.
- 173 Liz: Right. I mean you can find where it crosses this axis by putting $x = 0$.
- 174 Lorne: Uhmhm.
- 175 Liz: You can find where it crosses this one by putting $y = 0$.
- 176 Lorne: Right.
- 177 Liz: And if this is equal to 0, y is equal to 0, that means $x(x - a)$ is zero.
- 178 Lorne: Yes, yes.
- 179 Liz: So either x is 0 or x is a .
- 180 Lorne: Uhmhm, yes.
- 181 Liz: Right. So that's given you the two possible points
- 182 Lorne: yes
- 183 Liz: which you've ended up with.
- 184 Lorne: Of course, of course, yeah, uhmhm.
- 185 Liz: Umm, .. the second question is 'does this graph do for any value of a ?'
- 186 Lorne: ... Umm negative values of a equals, yes it will because if you're saying that $y = 0$, then the process that you've just said
- 187 Liz: Uhmhm
- 188 Lorne: should work for, even if it's negative.
- 189 Liz: Right. But what would be graph look like if a was negative?
- 190 Lorne: Then the upcoming side of the parabola would go through $(0, a)$. It wouldn't be downcoming.

- 191 Liz: Right, can you do one, do one on there (*indicating the paper*), to show me what you mean.
- 192 Lorne: So if it's negative a , go through and up that way.
- 193 Liz: Uhmhm. Okay. Right
- 194 Lorne: I don't think I've (*inaudible*)
- 195 Liz: I think umm I'm going to choose one for you now.
- 196 Lorne: Okay.
- 197 Liz: That one. (*I give him question 4 which reads 'Find, in terms of a and b , the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$*)
- 198 Lorne: Hmm. Right. .. $x + 2y - 4 = 0$ is the same as mmm .. right, $x + 2y = 4$ (*writes $x + 2y = 4$, $x = 4 - 2y$, $x - 4 = -2y$*)(*laughs*)
- 199 Liz: How many different ways can we write this equation?
- 200 Lorne: Right. (*writes $\frac{1}{2}x - 2 = -y$*) $-2 = -y$, which is,yes, Uhmhm one
- 201 Liz: Would you rather have it as $y = ?$
- 202 Lorne: Yes. So it's $y =$ minus the whole lot isn't it?
- 203 Liz: Yes.
- 204 Lorne: $y = -\frac{1}{2}x + 2$
- 205 Liz: Right.
- 206 Lorne: Which is, which is, which is, $x, 1, 2, \dots$ (*he begins drawing a coordinate graph to show the line*)
- 207 Liz: uhuh minus $\frac{1}{2}$
- 208 Lorne: So when that reaches a point 3 then that is 6, no, yes, yes. It's one and a half.
- 209 Liz: From there to there it's only going up one isn't it? (*I am indicating the point where the y -coordinate is 2 and where it is 3*)
- 210 Lorne: Yes. I don't know if it's a realistic scale but umm, ...($2, 1$), yes $6, -6$ even. (*writes -6 on the x -axis below the point where $y = 3$*)
- 211 Liz: No it would be -6 if it was going up 3.

- 212 Lorne: Oh, no, of course, yes, yes.
- 213 Liz: But it's only going up one.
- 214 Lorne: Umm it's - 2 .
- 215 Liz: Yes.
- 216 Lorne: Yes.
- 217 Liz: What about the point over here where it crosses the x -axis?
- 218 Lorne: Where it crosses the x -axis will be when $y = 0$ (*writes $0 = -\frac{1}{2}x + 2$*)
...which is,... is 4 anyway isn't it?
- 219 Liz: Hmm.
- 220 Lorne: I don't need to work that out. It's 4. Hmm.
- 221 Liz: Okay.
- 222 Lorne: Oh you want me to do
- 223 Liz: There's the small matter of the question here.
- 224 Lorne: Right, in terms of a, b the foot of the perpendicular to the point (a, b) to the line, to this line. Well the gradient of it is going to be 2 for the perpendicular line. perpendicular line are they, are they saying give the coordinates of (a, b) or the grad, the actual equation of the line ab of what it's, of what it's on it's perpendicular?
- 225 Liz: They're asking you for the coordinates of the foot of the perpendicular. Do you know what that means?
- 226 Lorne: It means it could be anywhere on this line. It is that
- 227 Liz: Yes, it is the foot of the perpendicular from the point (a, b) .
- 228 Lorne: So if (a, b) 's here, so the perpendicular's there?
- 229 Liz: No. (a, b) can't be on the line because if it's on the line then there isn't a perpendicular from it to the line.
- 230 Lorne: Oh.
- 231 Liz: So it can be anywhere that isn't on the line.
- 232 Lorne: If it's underneath it
- 233 Liz: yes
- 234 Lorne: then the foot of it is here (*indicates a point on the x -axis*)

- 235 Liz: No, the perpendicular goes from the point to the line, but it makes a right-angle with the line.
- 236 Lorne: Right. So it's back there.
- 237 Liz: Yes.
- 238 Lorne: So this is (a, b) , now I've got to find this point here?
- 239 Liz: Yes. That's it.
- 240 Lorne: Right. (x, y) they've got them in terms of a, b so should equal, umm, ... find that, .. well the gradient is 2, it's gradient, so it's going to be, yes, yes, yes. It's umm $b = 2a + c$, (*writes this*) where c is, it's $y = mx + c$.
- 241 Liz: Uhmhm.
- 242 Lorne: Which, to get c , $c = b - 2a$ (*writes this*) .. umm I don't need to know c or do I? I do, I do. Now I need, I need to get the point, I need to get something which fits these two coordinates that satisfy both
- 243 Liz: Uhmhm
- 244 Lorne: both equations.
- 245 Liz: A minute ago you were saying I don't need c , oh yes I do need c .
- 246 Lorne: No, I mean I was thinking because to try and find the, the umm, the equation of that line, you only know the gradient's 2
- 247 Liz: Uhmhm
- 248 Lorne: and you've got the points b and a so umm you could really put $b = 2a + c$
- 249 Liz: yes
- 250 Lorne: and then when you've found what c is
- 251 Liz: yes
- 252 Lorne: hmm.
- 253 Liz: What were you doing here?
- 254 Lorne: Well that's, that's what c is so
- 255 Liz: yes
- 256 Lorne: that's the same .. but I don't really need to use that because, .. it could be $y = 2x + (b - 2a)$ (*writes this*). for the equation of the line.

- 257 Liz: Uhmhm.
- 258 Lorne: .. So, like a simultaneous equation or something.
..... Umm - Pass! - umm.
- 259 Liz: What have you got here? What's this equation? (*pointing to the equation $y = 2x + (b - 2a)$*)
- 260 Lorne: That's the equation of the line.
- 261 Liz: Which line?
- 262 Lorne: The perpendicular line.
- 263 Liz: Okay. So what was this about simultaneous?
- 264 Lorne: I don't know because if I want to satisfy that into two simultaneous equations
- 265 Liz: yes, looks hard?
- 266 Lorne: Well if I'm going to do that, I mean I'm not going to get in terms of a and b or b and a or whatever, (*sneezes*) excuse me, ... I don't like simultaneous equations.
- 267 Liz: (*Laughter*) Well I'll do a deal with you. You tell me which two equations you want solved simultaneously, I'll do it.
- 268 Lorne: These two.
- 269 Liz: Okay. I've got $y = -\frac{1}{2}x + 2$ and $y = 2x + b - 2a$.
- 270 Lorne: Uhmhm.
- 271 Liz: Umm, $-\frac{1}{2}x + 2 = 2x + b - 2a$
- 272 Lorne: Uhmhm.
- 273 Liz: Get the x s together. $\frac{3x}{2} = b -$, erh no that's not right is it? $\frac{-5x}{2} = b - 2a - 2$ don't like negatives, don't like fractions, $5x = -2b + 4a - 4$. What is it I want to get from this?
- 274 Lorne: I don't know, I've just umm thought that you might, ohh, umm right hmm, by doing that I've got someone's having a fire,
- 275 Liz: yes
- 276 Lorne: never mind, umm, no.

- 277 Liz: What's, .. you wanted me to solve these two equations simultaneously.
- 278 Lorne: Uhmhm.
- 279 Liz: What would you, what do you find if you do that, .. what am I after?
- 280 Lorne: Umm I've done simultaneous equations to umm because the question asked you to do a simultaneous equation.
- 281 Liz: Oh no it doesn't. Not this one.
- 282 Lorne: Oh no, no. This doesn't but usually circumstances umm hmm, no, I'll pass on the question.
- 283 Liz: *(laughter)*
- 284 Lorne: Give away four marks that
- 285 Liz: You're not allowed to give up. Umm we've got the equation of this line
- 286 Lorne: yes
- 287 Liz: and the equation of this line.
- 288 Lorne: Uhmhm.
- 289 Liz: And you're solving them simultaneously in order to find out where they meet, I think.
- 290 Lorne: Yes, that's the one Oh, because, if umm you divide that by 5, *(writes a fraction line and a 5 under the line $-2b + 4a - 4$)* x and afterwards that's coordinate x .
- 291 Liz: Uhmhm.
- 292 Lorne: Isn't it? Umm, I was actually thinking of changing these to something else because it was getting a bit awkward but, that's coordinate x and then you can work out y .
- 293 Liz: Uhmhm. .. Have we answered the question?
- 294 Lorne: Umm we will do when I work, well then I'll just put that, yes, because that's the x -coordinate, and then you work out the y -coordinate.
- 295 Liz: Right. You want me to work out the y -coordinate?
- 296 Lorne: Oh, I'll give it a go, I'll give it a go. Umm $-\frac{1}{2}$, no sorry.

- 297 Liz: $-\frac{1}{2}$ of $x + 2$ (*writes* $-\frac{1}{2}(\frac{-2b + 4a - 4}{5} + 2)$, so I can divide all of that top by 2, to get $b - 2a + 2$ (*gets the equation* $y = \frac{b - 2a + 12}{5}$))
- 298 Lorne: That's the y -coordinate, which makes umm, which makes the answer $\frac{-2b + 4a - 4}{5}$.. and then $\frac{b - 2a + 12}{5}$
- 299 Liz: Okay. This question talks about point (a, b) .
- 300 Lorne: Hmm.
- 301 Liz: And it asks you to give an answer in terms of a and b .
- 302 Lorne: Uhmhm.
- 303 Liz: Umm, what's, what's going on then? Tell me about this point (a, b) and this point that we've found.
- 304 Lorne: Well once you've got a point (a, b) like if that's $(1, 2)$, then afterwards you can work out what that is because you can, you've just got a and b in there, so you can work out the coordinates of where it meets then.
- 305 Liz: Right. Umm so could you work out the coordinates of the foot of the perpendicular from the origin to that line?
- 306 Lorne: From the origin to that line?
- 307 Liz: Yes.
- 308 Lorne: Umm, umm, it's, ... well yes you can because you get your coordinate (a, b) , even if it's up here, then afterwards it's a along and b , so it's just use Pythagoras there, and you work out, you're saying the length of that from the origin, the origin's here. Yes?
- 309 Liz: No. Listen again.
- 310 Lorne: Okay I'll listen.
- 311 Liz: Can you find the foot of the perpendicular
- 312 Lorne: Uhmhm
- 313 Liz: from the origin to that line?
- 314 Lorne: from the origin to that line?
- 315 Liz: Uhmhm.
- 316 Lorne: Does it work out a triangle?

- 317 Liz: Can you draw in the perpendicular from the origin to that line?
- 318 Lorne: The perpendicular from the origin, to this line?
- 319 Liz: Yes.
- 320 Lorne: No, (*draws in the appropriate line*) is that what you're saying?
- 321 Liz: Yes.
- 322 Lorne: Uhmhm.
- 323 Liz: Now if we wanted to work out the coordinates of that point
- 324 Lorne: Uhmhm
- 325 Liz: can we use the working that we've done in this question to help?
- 326 Lorne: Yes you can. You can because it's going to be $-\frac{4}{5}$ - oh it can't be $-\frac{4}{5}$
(*he sees from his diagram that the x-coordinate cannot be $-\frac{4}{5}$*).
- 327 Liz: Hmm, I think we've got some working wrong somewhere.
- 328 Lorne: Hmm. $\frac{12}{5}$ might seem reasonable.
- 329 Liz: Umm $\frac{12}{5}$ is more than 2, isn't it?
- 330 Lorne: Well it won't, it won't, no, it won't do. Yes.
- 331 Liz: Yes. When I say we've got some working wrong here
- 332 Lorne: Uhmhm
- 333 Liz: I'm the one that did the working.
- 334 Lorne: Right. I see.
- 335 Liz: [*laughter*]
- 336 Lorne: Okay. We should be
- 337 Liz: So, how do you know it's wrong?
- 338 Lorne: Because that gives $-\frac{4}{5}$ and that means it's the x-coordinate
- 339 Liz: right
- 340 Lorne: is negative and it can't be.

- 341 Liz: What do you think it should be?
- 342 Lorne: It should be umm well the y -coordinate has got to be less than 2, and the x -coordinate has to, is going to be .. around 1-ish.
- 343 Liz: Right. So I could believe that it should be $\frac{4}{5}$ ths and $\frac{8}{5}$ ths.
- 344 Lorne: Yes, you could, you could. It could be. It could well be.
- 345 Liz: So where have I gone wrong? Oh, well that's the obvious, the first one. When I multiplied through by a minus here
- 346 Lorne: h m m
- 347 Liz: I changed the first two signs but I didn't change that one.
- 348 Lorne: Which means that yes
- 349 Liz: So when we come to here, that should have been plus which makes that minus
- 350 Lorne: minus, which makes it 8. Uhmhm.
- 351 Liz: Right.
- 352 Lorne: Yes,
- 353 Liz: It's a good idea to check it. *[laughter]* Okay. I think that's plenty. Thank you very much.
- 354 Lorne: Right.

- 1 Liz: Umm, could you start with that one. (*I hand him question 2 which reads 'The point (a, b) is equidistant from the x-axis and the point (1, 2). Find an equation linking a and b'*)
- 2 Neil: I've done this before - not this particular question but quite a similar one, 'The point (a, b) is equidistant from the -'
.....
- 3 Liz: What's the pause about?
- 4 Neil: The pause is about, although I did this before it was with Pythagoras because you know this length and that length, and you know, so you can find an equation to find that length but
- 5 Liz: ah hmm
- 6 Neil: umm but I'm trying to find it, because it's equal to 1 there's no way of labelling it, that length.
- 7 Liz: No way of labelling which length?
- 8 Neil: This length here.
- 9 Liz: Right.
- 10 Neil: I'm not sure of a way of labelling it.
- 11 Liz: Umm, well, have a look back at the question.
- 12 Neil: this distance
- 13 Liz: What distance are you working on now?
- 14 Neil: I'm just working on that length, because if they're equidistant, this one's easier, I mean from that length is to that length.
- 15 Liz: Ah hmm.
- 16 Neil: Is equal that's equal to , that equals that (*inaudible*) (*Neil has now got to the equation $a^2 - 2a + 5 = 4b$*)
- 17 Liz: Okay. Umm, you said you'd done a question like that before.
- 18 Neil: Yes.
- 19 Liz: What do you mean by a question like that?
- 20 Neil: Virtually the same question with 2 point, umm, is eq, you've got an unknown point and they're equidistant from the x-axis. That was in umm line geometry.

- 21 Liz: Right.
- 22 Neil: P1.
- 23 Liz: Okay, umm, this answer that you've got
- 24 Neil: yes
- 25 Liz: umm, what does it tell you? Why did we want to know that?
- 26 Neil: Umm why did we want to know that? It shows you, well it gives you a range of solutions for a and b .
- 27 Liz: Ah hmm. Umm can you link that with anything else? Does that equation represent anything?
- 28 Neil: Well it's, it's a quadratic so it's going to give you at least two answers two solutions.
- 29 Liz: .. Two solutions to what?
- 30 Neil: To where the point can be, the point (a, b) can be.
- 31 Liz: Ah hmm. Umm let's have a slightly larger diagram of that. So here's the x -axis and the point $(1, 2)$ about here. (*I draw a diagram*) Umm
- 32 Neil: It's going to, it's going to be, it's going to be something umm I don't know, I don't know what the word is but I see it regularly. It's not gonna be, not going to be another random point, because there's two solutions, there's going to be two answers. It's going to be equidistant from there and the x -axis. So the point's going to be umm, I don't know, maybe a reflection there or a reflection there or something similar.
- 33 Liz: Ah hmm. Can you
- 34 Neil: Umm, but it's not
- 35 Liz: Can you see if you can mark in a point that fits the conditions of the question, equidistant from $(1, 2)$
- 36 Neil: $(1, 2)$
- 37 Liz: and the x -axis.
- 38 Neil: $(1, 2)$'s there. Is this supposed to be $(1, -?$
- 39 Liz: That's supposed to be $(1, 2)$.
- 40 Neil: Yes, all right, so, yes I mean if you just go across the same distance
- 41 Liz: ah hmm

- 42 Neil: if you go across 2, there
- 43 Liz: right, so that's the point (3, 2)?
- 44 Neil: Yes.
- 45 Liz: Okay. Any more?
- 46 Neil: Oh, well it works equally if you go that way.
- 47 Liz: Ah hmm.
- 48 Neil: So you're gonna have point minus, (-1, 2). And that's gonna be -
- 49 Liz: Yes. So are those the only two?
- 50 Neil: Umm,perpendicularly, you're just working like that, and it becomes harder once you get past the x -axis because this distance is always going to be greater.
- 51 Liz: Right. So you can't go underneath the x -axis?
- 52 Neil: No. You probably can't. Umm
- 53 Liz: ... What's the lowest point you could get to?
- 54 Neil: .. I suppose 2. 2 away from that, - zero, because once you get past that you're starting to get, you're having an impossible situation where you're adding the distance between the x -axis and the point is a certain amount, and the distance from the point to (1, 2) is that amount and, and 2 again
- 55 Liz: ah hmm
- 56 Neil: so it's always that much further away. Umm no, looks messy. Probably shouldn't go below the x -axis.
- 57 Liz: Okay. Umm what about the point here, (1, 1).
- 58 Neil: Yes, that's good enough, don't know why I didn't think of that. Yes, exactly in the middle so that, so that's going to work isn't it? As long as you stay on that line of symmetry you can move either way. Oh no it's not quite like that is it? No, because it's perpendicular every time umm no, .. that would work because the
- 59 Liz: ah hmm
- 60 Neil: same distance both sides
- 61 Liz: As you say if you, if you do move this way
- 62 Neil: yes, I was thinking of it being like that.
- 63 Liz: Yes.

- 64 Neil: But it's not like that at all.
- 65 Liz: Hmm, so you'd have to move up a bit wouldn't you?
- 66 Neil: Yes. I suppose you're going to start, start drawing circles or a sort of loci of points. *(he draws in a circle going through (-1, 2), (1, 1) and (3, 2))*
- 67 Liz: Ah ha.
- 68 Neil: Of all the different points on there, on this equation
- 69 Liz: Right. Well I'm certainly happy that we would have something like this *(I draw an arc from (-1, 2), through (1, 1) ending at (3, 2))*
- 70 Neil: yes
- 71 Liz: okay, I'm not sure what happens after that. Umm
- 72 Neil: That's just a parabola isn't it? So you just keep, if that's a parabola then you probably think that's going to, the points are just going to go out like this. *(he sketches more of the parabola)*
- 73 Liz: ... Ah hmm.
- 74 Neil: .. I don't know, it means that this hasn't got any solution to it if that's the case, then there's no, there's no, there's no answer to when this equals nothing, which makes sense because it's
- 75 Liz: ah hmm, could you check whether there are any roots for that?
- 76 Neil: *(he begins by writing $4 - (4 - 2.5)$)*.....No, sorry that's wrong. *(he writes a new expression for b, $\frac{a^2}{4} - \frac{a}{2} + \frac{5}{4}$ and calculates the discriminant.)*..... there aren't any roots, because that just goes to 1, so you've got $\frac{1}{4} - \frac{5}{4}$ which is a negative number
- 77 Liz: ah hmm
- 78 Neil: so you don't get any roots which is what you are saying there
- 79 Liz: Right, so, umm, can we come back to the question I asked you a few minutes ago. What does that equation do for us?
- 80 Neil: It tells us all the points umm, .. it just gives us a set of points at which this is true.
- 81 Liz: Hmm, which as you say is called the locus.
- 82 Neil: Yes.
- 83 Liz: Umm I don't think you've done anything about locus at school, have you?

- 84 Neil: We've done drawings at some stage
- 85 Liz: Right, right, so that's where you've picked the word up from.
- 86 Neil: Yes I think so.
- 87 Liz: Okay. So that umm equation is actually the equation of the parabola
- 88 Neil: yes
- 89 Liz: as you say on which points must lie if they fit these conditions.
- 90 Neil: Yes
- 91 Liz: Okay. I think that'll do for that one. That's rather boring (*referring to another question which I am deciding not to give him*)
- 92 Neil: It's interesting because we've never took it that far.
- 93 Liz: Hmm.
- 94 Neil: I've had that question twice actually. I had it when we were doing it in, earlier this year and I remember it coming up in the exam as well and, you know, and I never thought about it much more than 'that's how you do it to find the equation'.
- 95 Liz: Yes, well the, the business of interpreting it as a locus really comes on P2 so
- 96 Neil: yes
- 97 Liz: the teacher wouldn't have pushed you any further with it. That's what it actually represents, that's why you were doing it.
- 98 Neil: 'Find the equation of the tangent to the curve, $y = x^2 + 1$ which passes through the origin' (*he reads the next question and begins drawing a diagram showing the curve*) I like diagrams.
- 99 Liz: Hmm.
- 100 Neil: I think they always make things easier. There's got to be two (*he draws two tangents going through the origin*). For a start it says tangent but
- 101 Liz: ah hmm
- 102 Neil: there's gonna be two.
- 103 Liz: Nobody promised these questions were fair.
- 104 Neil: No, no, obviously not. .. I mean the obvious step is to differentiate umm if, if I knew some relationship about what sort of

gradient that (*indicating the tangent*) had, I could differentiate and say it equals that

105 Liz: ah hmm

106 Neil: .. and the question is, at what, at what point does it cross or .. because we know this point, that's 1. .. Because of, I don't know, we need an idea of where they cross. (*he writes $\frac{dy}{dx} = 2x$, $y = 2x$*)..... I know it's going to be, Oh that's it, yes, because it doesn't go, it goes through the origin so that's just - it's that because it goes through (0, 0).

107 Liz: Ah hmm.

108 Neil: So it's got no intercept.

109 Liz: Right. So,

110 Neil: Oh no that's wrong because that's the equation for the point, that's the equation for a function of a gradient, umm any value given a value x .

111 Liz: Ah hmm.

112 Neil: Right, try and understand what I'm doing. So, I need to, I need to, to find something that will tell me - (*reads*) 'the equation of the tangent', it's the problem that I've got, the reason I'm stuck is that I'm not sure .. a way of finding this point or that point (*he indicates the two points of contact of the curve with the tangents.*)

113 Liz: Ah hmm, yes.

114 Neil: You know, if I had either of these I could find the other point, and if I had the points umm it's easy enough to find the equation - if I had the point I could find the gradient.

115 Liz: Right. Let me make a suggestion then.

116 Neil: Yes.

117 Liz: Pretend you know what that point is.

118 Neil: Yes, so it will cancel out, right. (*he labels the two points (a, b) and $(-a, b)$*) okay, so if I know that one's - then the gradient, (*writes gradient = $2a$*) $2a$, yeah, right, that's the equation for this line, hopefully (*he has written $y - b = 2a(x - a)$, $y = 2ax - 2a^2 + b$*) no .. umm this has got this equation - it goes through (0, 0) so . $x = 0$, when $y = 0$, $x = 0$. So I've done erh this. now that - when $x = 0$, that expression equals 0 (*he indicates $2ax$*)

- 119 Liz: ah hmm
- 120 Neil: so does that (*he draws a circle around $-2a^2 + b$*). And so as far as that and .. no reason, there's nothing saying it should, but that there until, I don't know why I've got, why I've got that and why, I suppose you're just going to have to say that $2a^2 - b = 0$, very sticky. Well I don't know, don't know whether I'm going round in circles or not. That means I can replace the b there with $2a^2$ and then that will cancel out and give me $y = 2ax$ but that's possibly just happened because I replaced things back into each other. That's hardly, that's just replacing to each other
- 121 Liz: Ah hmm.
- 122 Neil: Umm, I mean, yes, I would have said that answered the equation but it doesn't, doesn't ... agree with it, doesn't make sense because, because they should equal nothing at that point
- 123 Liz: ah hmm
- 124 Neil: that's the point you know.
- 125 Liz: Yes. When you get umm an answer to this question, is it going to have as and bs in it? Umm the question is find the equation of the tangent, so would you expect your final answer to have as and bs in it.
- 126 Neil: Umm I would - in a sense what you're saying is that it's, does the equation of a tangent, is it related to where the point is and
- 127 Liz: that's not quite the same thing as the question I'm asking
- 128 Neil: okay, sorry
- 129 Liz: but it's the, also a worthwhile question.
- 130 Neil: Yes. Does it have as and bs in it. I mean, I would have thought it does because well I would have thought the question to be the same because whether it has as and bs in it was whether, whether as and bs make any difference at that point, if it
- 131 Liz: ah hmm
- 132 Neil: if as and bs aren't in it, then it wouldn't make a difference.
- 133 Liz: Yes, that's one way of looking at it. Umm here's another way of looking at it. Umm is the question as it stands answerable? Is there only, well are there only two tangents to that curve which go through the origin?
- 134 Neil: .. Yes. I would say from that curve there are only two tangents in that.

- 135 Liz: So if we were actually to sort of get a piece of graph paper and plot this graph on it,
- 136 Neil: yes
- 137 Liz: umm we could draw in the tangent which goes through the origin and we could find out what it's equation is, .. without having to put an a and b in the answer.
- 138 Neil: That's, .. yes but that was because you would be taking the values of a and b from the graph paper.
- 139 Liz: Umm yes, but I'm not making them up.
- 140 Neil: .. What do you mean?
- 141 Liz: I'm working out what a and b are as numeral, as numbers, numerical values.
- 142 Neil: Yes, so, it's the same because they would be, a and b represent the all the numbers you could choose, so,
- 143 Liz: ah hmm
- 144 Neil: I mean, for a start I haven't quite answered the question because I mean it's the tangent, so in that 'Find the -' no, no, the tangent, yeah. I don't, .. I don't like that from, from what I know that's, that's what it's derived, but -
- 145 Liz: Ah hmm.
- 146 Neil: This doesn't make sense to me. This doesn't, it's not right because
- 147 Liz: okay umm .. if you use all the information that's available to you, you can work out what the values of a and b have to be.
- 148 Neil: Yes.
- 149 Liz: Umm you've made a start on that
- 150 Neil: yes
- 151 Liz: and you've found that one condition controlling the values of a and b is that $2a^2 - b = 0$. That has to be the case in order for the tangent to go through the origin.
- 152 Neil: Yes.
- 153 Liz: But there is more information that you haven't yet used, and if you can work out what that is and use it, you'll be able to find actual values for a and b .

- 154 Neil: All those, (*inaudible*) I don't know where else to go.
- 155 Liz: .. Umm okay, what information have you used about the point (a, b) so far? What have you done?
- 156 Neil: Umm lies on the curve, and
- 157 Liz: where have you used that?
- 158 Neil: Well I don't know, I know, I was thinking we do have the relationship $y = x^2 + 1$ so .. yes I suppose you could use that. I was thinking that that, that's true (*writes $a^2 + 1 = b$*). Now which part do we use to solve it? I suppose you're saying, are you saying that $2a^2 = b$ and $a^2 + 1 = b$. Now is that going to solve, a^2 , so $2a^2 - a^2 - 1 = 0$, $a^2 - 1 = 0$ (*writes as he speaks*) which of course is very nicely symmetrical because you've got two points there anyway (*writes $(a + 1)(a - 1) = 0$*) So, $a = 1$ or -1 and therefore $b =$ (*writes $b = 2, 2$*) the same point each time - it does. That's very nice. so now I've got those, we can find the equations and (*writes $y - 2 = 2(x - 1)$, $y = 2x$*)
- 159 Liz: (*laughter*)
- 160 Neil: (*inaudible*)
- 161 Liz: I thought you might see the irony of this.
- 162 Neil: It's a great problem. (*he writes $y - 2 = -\frac{1}{2}(\)$*)
- 163 Liz: Where did you get the half from?
- 164 Neil: Um, I'm talking gibberish, thinking gibberish.
- 165 Liz: Ah hmm.
- 166 Neil: umm yes because they're not perpendicular that's why it's not a half.
- 167 Liz: Ah hmm.
- 168 Neil: (*inaudible*) .. a very neat problem.
- 169 Liz: Ah hmm. Umm you had the erh equation $y = 2x$ in your second line.
- 170 Neil: Hmm, yes but,
- 171 Liz: umm, so why couldn't you have said it there?
- 172 Neil: Because that's the equation for, that tells me what if I've got a value of x , what the gradient is at the point.

- 173 Liz: Right.
- 174 Neil: Not an equation for the line.
- 175 Liz: Okay.
- 176 Neil: I got that from differentiating the original equation and not from .. umm, not from the knowing where the points were
- 177 Liz: Can you see why it turns out to be the right answer anyway?
- 178 Neil: Umm, .. not straight off, I mean, is there ... , yes, I can see how it happens because the gradient's 2 and it goes through (0, 0).
- 179 Liz: Ah hmm.
- 180 Neil: So
- 181 Liz: Umm
- 182 Neil: It's quite straightforward if you can see it - I suppose it's easy.
- 183 Liz: Also the point at which the tangent touches happens to be the point where $x = 1$. So
- 184 Neil: How do you know that?
- 185 Liz: Well I don't, I didn't know beforehand but
- 186 Neil: yes
- 187 Liz: that happens to be the answer. Umm I'm not saying that you could have seen this from the beginning. I'm saying now that we know how we work it out, we can see why this was right. Umm
- 188 Neil: I think I can, I don't know, perhaps you know, I mean, I was on to it, I was suggesting there that that I knew the relationship had to be the fact that it had to be zero there
- 189 Liz: ah hmm
- 190 Neil: so I suppose I could have, see now I could have jumped to it.
- 191 Liz: Hmm, well, I don't think it's very easy to jump to it. Okay. Umm, could you describe to me now the stages that you've gone through in getting to that tangent equation.
- 192 Neil: Well first as soon as they mention tangent
- 193 Liz: right
- 194 Neil: it means to me start to differentiate - and I did that and then you suggested that I choose values for a and b so that I - I chose a and b

for the right hand one and then I saw that the other one would be negative a and b

195 Liz: ah hmm

196 Neil: and then that means if you apply x coordinate, you know the gradient's $2a$, and the formulas we've been told for umm straight lines, $y - y_1 = m(x - x_1)$ and then just apply that and it will tell you a straight line formula. That didn't, I didn't like that because it had to be, it had to go through $(0, 0)$ and umm there was a, yes, $-2a^2 + b$ on the end and when $x = 0, y = 0$ so I wrote down $2a^2 - b = 0$, I knew that was right. And then I didn't seem to go anywhere and then I looked back at the, well, as you suggested, I looked back at the equation, the original statement to the equation, (*inaudible*) that before and you get a chance for another relationship between a and b . So I looked back and saw that, as soon as I saw what that relationship was you could see the answer because

197 Liz: right

198 Neil: you can get a sum of, sum of squares, difference of squares, so it would just give me two answers either side of the axis.

199 Liz: Okay. Umm I'm not going to ask you to do any of the other questions, but, umm there's a question which John and I gave to some sixth formers in Suffolk when we were down there a few months ago, which is very similar to the one you've just done

200 Neil: right

201 Liz: umm except that what we were asking them was not about the curve $y = x^2 + 1$ it was about the curve $y = e^x$. Umm have you come across e^x ?

202 Neil: Last lesson.

203 Liz: Right, okay. So you

204 Neil: I haven't

205 Liz: You won't have umm

206 Neil: seen it before

207 Liz: differentiated it yet?

208 Neil: I do know what the differentiation

209 Liz: you do know that the derivative is e^x .

210 Neil: Yes, yes, that's just a fact. I don't know how to get there.

- 211 Liz: Oh that's all right. That doesn't matter. Umm and I want to show you the sort of working they did and see what you make of it. Umm they did something like this. They wrote this down and what they're trying to find is the equation of a tangent that goes through the origin. So they wrote down $\frac{dy}{dx} = e^x$ (I write down $y = e^x, \frac{dy}{dx} = e^x$)
- 212 Neil: h m m
- 213 Liz: and then they said the equation of tangent is that. (I write 'equation of tangent $\Rightarrow y = e^x x$ '))
- 214 Neil: Oh yes, because it goes through the origin.
- 215 Liz: Ah h m m.
- 216 Neil: Yes.
- 217 Liz: Umm now what was the next step?
- 218 Neil: Your memory is very impressive!
- 219 Liz: Oh yes, they said umm at the point of contact, where the tangent touches the curve
- 220 Neil: yes (I write 'At the point of contact $e^x = e^x x$, $\therefore x = 1, y = e$ ') They might have said that because they know the value, at that point, the value of - yes I see that because ..
- 221 Liz: I think I might have missed out a line of their working actually. If I put that in it might help you and they said $y = e^x$ and $y = e^x x$.
- 222 Neil: why?
- 223 Liz: Well that's the equation of the curve, (pointing to $y = e^x$)
- 224 Neil: yes
- 225 Liz: that's the equation of the tangent (pointing to $y = e^x x$)
- 226 Neil: Yes.
- 227 Liz: And where they cross
- 228 Neil: yes
- 229 Liz: they're gonna both be true, so they put those two together like this.
- 230 Neil: Ah yes, yes.
- 231 Liz: Okay. And then the conclusion was $x = 1$.

- 232 Neil: Umm ... yes, yes, yes. I can see that that would work.
- 233 Liz: They're saying that the point where the, not a very good diagram I'm afraid (*I draw a diagram showing the curve and the tangent*) this is the point where $x = 1$. And in fact, umm, I've mislead you about what the question was they were being asked to find the point of contact not the equation of a tangent.
- 234 Neil: Yes.
- 235 Liz: And so that was umm $y = e$ that was their answer.
- 236 Neil: Yes.
- 237 Liz: . Umm have another look at it. It, the answer is correct, so I'm not asking you to check their working, but have a look at their reasoning and see what you think of it.
- 238 Neil: Seems to make sense to me.
- 239 Liz: Right. Okay. Umm, it does make sense, and they get the right answer and so on. Umm this is a bit odd (*I point to 'equation of tangent $\Rightarrow y = e^x x'$*)
- 240 Neil: Yes, that's a bit of a an assumption and I only agree with you because of the question we just did.
- 241 Liz: Yes.
- 242 Neil: Being caught out last time -
- 243 Liz: Don't you think that umm what they've done here is rather like your saying that the equation of the tangent is $y = 2x$?
- 244 Neil: Well it seems as if it was. Yes
- 245 Liz: Hmm, yes.
- 246 Neil: Yes, I think that's right, but umm .. it's, it is but you know that the equation is, if the gradient, the equation I'd always use for umm, for a line, that we're told is equal to ..(*he writes $y - y_1 = m(x - x_1)$*) yes
- 247 Liz: ah hmm
- 248 Neil: you've got that and you know that the gradient is e^x
- 249 Liz: ah hmm
- 250 Neil: and your point, .. umm so if you know that, if you're choosing, if you take, all you need is one point you know, and one point you do know is that (*writes (0, 0)*)

- 251 Liz: yes
- 252 Neil: so you, you get, *(he writes $y = e^0(x - 0)$)* yes that's why I don't like that because it starts to .. yes $x = 0$ there. I don't know if that applies or anything *(he writes $y = ex$)*..... because then you get $y = e^x$ not e^xx .
- 253 Liz: Hmm.
- 254 Neil: Does this x on the e take values of x ? Yes it must do, I mean it's not, .. and that's a completely different equation. Now is that wrong or is it just different .. because um - That's fine, that disappears, that disappears *(he rewrites ex as $1x$)*.. you just get e^x but basically, that disappears and it's just 1.
- 255 Liz: Hmm.
- 256 Neil: Which is $y = x$ Possibly.
- 257 Liz: Umm, you put a 0 in here *(indicating the zero in e^0)* because of this 0 *(indicating the zero in the bracket $(x - 0)$)* did you?
- 258 Neil: Yes that's the bit I'm not sure about, don't know, is that feasible?
- 259 Liz: Umm, the e^x that you're putting in there is the gradient.
- 260 Neil: Yes.
- 261 Liz: umm the gradient of what?
- 262 Neil: .. The gradient of a tangent.
- 263 Liz: Ah hmm, which we derived from the gradient of the curve.
- 264 Neil: Yes, yes, yes, that's right, yes, because that's not the gradient of the tangent, it's the gradient of the curve, at any point.
- 265 Liz: Hmm, and what we actually want is the gradient of the curve at that point *(indicating the point of contact)*
- 266 Neil: Yes. Hm.
- 267 Liz: And at the moment we don't know where that point is.
- 268 Neil: Yes well we will, I suspect it might be 1, but - Right.
- 269 Liz: Well we know that it's one, but at the moment we don't. Umm so there's a big query over what you do put there.
- 270 Neil: Yes.
- 271 Liz: Not zero, .. because, I mean, that point certainly isn't the origin. Umm on the other hand, if you put x in

- 272 Neil: But as soon as you don't start putting points that aren't zero then these values are going to have to come into play. They start making a difference.
- 273 Liz: Umm well the value that goes there (*indicating the m in $y - y_1 = m(x - x_1)$*) is connected with that point (*the point of contact*). These two values (*indicating the x_1 and y_1*) are they also of that point?
- 274 Neil: Yes because they're connected with the value $y = e^x$.
- 275 Liz: Say that again.
- 276 Neil: These points, you know this point and that point, between this.
- 277 Liz: Hmm. Right. But what have they got to do with x_1, y_1 here.
- 278 Neil: Well x_1 there, that's this point (*the point of contact*).
- 279 Liz: Right.
- 280 Neil: So - no -
- 281 Liz: That's not what you used the first time.
- 282 Neil: No.
- 283 Liz: When you moved from that line to that line (*from $y - y_1 = m(x - x_1)$ to $y = e^0(x - 0)$*) you used $(0, 0)$ as (x_1, y_1)
- 284 Neil: yes, because it's a point I knew.
- 285 Liz: Yes.
- 286 Neil: But if you look at that now, you've got it again, the gradient, the equation for the gradient on there only applies at that point.
- 287 Liz: Ah hmm.
- 288 Neil: So it's of no use unless you know the point.
- 289 Liz: Hmm. Right. Well umm you have again to suppose that you do know it
- 290 Neil: yes
- 291 Liz: which is actually what these people have done, umm, but they've called it x , umm and then there's a, there's a high risk of getting this x confused with that x . (*pointing to the two x s in $y = e^xx$*)
- 292 Neil: which I did. They're not the same.
- 293 Liz: No they're not. This is the x at that point

- 294 Neil: yes
- 295 Liz: this is the x all the way along the tangent.
- 296 Neil: Yes.
- 297 Liz: Umm .. but in fact they get away with it because when they get to this line (*the line $e^x = e^x x$*), and they cancel out those two,
- 298 Neil: hmm
- 299 Liz: umm and it's only this x that is left, so there is no umm risk of getting it confused with that one any more.
- 300 Neil: Yes, (*inaudible*). Then we've got (*writes $y - b = e^x(X - a)$*).....
- 301 Liz: What's that little x ?
- 302 Neil: This (*pointing to the capital x*)? It's just
- 303 Liz: No this one.
- 304 Neil: .. That is the value at the point a , , oh yes that's right so (*writes $y = e^a X - ae^a + b$*) Make sure I've done it right, and then you get again, and then you've got an equation for this line and then you have to know that $ae^a - b$ is equal to nothing (*writes $ae^a - b = 0$*) and we also know that .. $b = e^a$ (*writes $b = e^a$*) (*inaudible*)
- 305 Liz: (*inaudible*) - is what?
- 306 Neil: It's, it's sort of, well it's the same thing twice.
- 307 Liz: Yes (*Neil writes $ae^a - e^a = 0$, $e^a(a - 1) = 0$*)
- 308 Neil: I mean .. yes I can solve that without much problem. I don't know if its right, I don't think it's a very nice way of doing it.
- 309 Liz: Hmm.
- 310 Neil: Umm and you can, you can get rid of that one by just sort of throwing it in the hole or that by just throwing it - trying to divide through by it.
- 311 Liz: Right. You can only throw them in the hole if they can't be zero. And if they could be zero they bounce out again.
- 312 Neil: Is that what happened to you before?
- 313 Liz: Well umm if you divide through by zero,
- 314 Neil: hmm, most teachers scream at us if we do that

- 315 Liz: yes that's right because umm supposing - well let's take this equation for example (*referring to $e^a(a - 1) = 0$*)
- 316 Neil: well erh
- 317 Liz: that could be zero, yes? $a - 1$ could be zero. So umm I get a solution to this equation because that full side then is zero (*pointing to $e^a(a - 1)$*)
- 318 Neil: h m m
- 319 Liz: and that side's also zero.
- 320 Neil: Yes.
- 321 Liz: So if I throw this in the hole then I've lost a solution.
- 322 Neil: Yes that's right so I mean a obviously can be 1
- 323 Liz: ah h m m
- 324 Neil: which is what they've found
- 325 Liz: yes
- 326 Neil: yes. And
- 327 Liz: What about the other one?
- 328 Neil: You need to put $e^a = 0$ and that means that .. , erh can't really do that can you?
- 329 Liz: No. Have a look at the graph.
- 330 Neil: I mean there's no way it crosses.
- 331 Liz: That's right. So e^a can't be zero so you can throw it into the hole.
- 332 Neil: Yes, so you just get $a = 1$, which is equal to x
- 333 Liz: h m m
- 334 Neil: at this point.
- 335 Liz: Yes.
- 336 Neil: Which is what we found before.
- 337 Liz: Yes.
- 338 Neil: Now have I proved it in any better way than what - or any sounder way than the way they did. Is that, is that an improvement, have I?

- 339 Liz: Your notation is better.
- 340 Neil: Yes, it's from practice that's all. And what
- 341 Liz: You've actually done exactly the same that they did. You've just expressed it in a different way.
- 342 Neil: Yes. Umm have I made any grounds which .. erh anything different to what they did? Was it progression - was it something different?
- 343 Liz: Umm yes, I think so because they used x to stand for two different things.
- 344 Neil: Hmm.
- 345 Liz: Umm and I said before they got away with it because these things cancelled out
- 346 Neil: yes
- 347 Liz: that they weren't affected by the x . You can imagine what would have happened umm if you'd been doing the same sort of thing - that one that I gave you originally. Umm first of all when you get to this point (*pointing to the line 'equation of tangent $\Rightarrow y = e^x x$ '*) umm you would actually have said well the gradient is $2x$, umm so if I do gradient - $y = \text{gradient times } x$ which is the equation of a line through the origin, $y = mx$, then that would have been $y = 2x$ times x .
- 348 Neil: ... I wouldn't have - yeah but that's -
- 349 Liz: You, you wouldn't have said that, no, but that's the
- 350 Neil: I would have done $2a$
- 351 Liz: that's the same step that they've done.
- 352 Neil: And that
- 353 Liz: $y = \text{gradient times } x$
- 354 Neil: that's wrong
- 355 Liz: hmm, well it's dangerous, because in that case you would have got $y = 2x^2$
- 356 Neil: yes
- 357 Liz: it's obvious to anyone that that's not a tangent.
- 358 Neil: Yes.

359 Liz: Umm but because these things weren't easily combinable
360 Neil: yes
361 Liz: it didn't become obvious that that wasn't the tangent.
362 Neil: Yes.
363 Liz: Anyway, I think that's enough of that.

- 1 Liz: Have you chosen one?
- 2 Paul: Yes, this one. (*Find the equation of the line with gradient M passing through the point (p, q)*)
- 3 Liz: Okay, erh, what's it about?
- 4 Paul: It's about finding the equation of a line
- 5 Liz: Uhmhm
- 6 Paul: which you don't know it and it's in general terms.
- 7 Liz: Right.
- 8 Paul: It's like finding a general equation, an equation of the line.
- 9 Liz: Okay.
- 10 Paul:You want me to do it?
- 11 Liz: Yes.
- 12 Paul: Right. Umm can I write on these or not?
- 13 Liz: Yes, please do.
- 14 Paul: Oh right. Okay. First of all if I remember correctly the formula is something like that (*writes $y - y_1 = m(x - x_1)$*) or there is a formula of something like that (*writes $y = mx + c$*), yes?
- 15 Liz: Uhmhm.
- 16 Paul: Which then go to, (*writes $y - q = m(x - p)$*)..... like so, substituting p and q for y_1 and x_1
- 17 Liz: Uhmhm
- 18 Paul: then I could rearrange that, multiply that out first. (*writes $y - q = mx - mp$*) Put it all on one side over there (*writes $mx - mp - y + q = 0$*).....
- 19 Liz: I'm just going to come and sit on the other side then I'll be able to see what you're writing. Okay.
- 20 Paul: The only problem is that gives me .. a lot of letters
- 21 Liz: h m m
- 22 Paul: but not (*inaudible*). It's not in a form where I've got $y =$, oh I don't know. Usually I get a ' $y =$ ' form. Usually a y over there. Umm I'm trying to get $y = mx + c$.

- 23 Liz: Uhmhm.
- 24 Paul: So you've got $y = mx$, there's the mx there (*writes $y = mx - mp + q$*). So that (*pointing to $-mp + q$*) is going to equal c but I don't know how. The p add the gradient times the two coordinates. No, gradient times the x coordinate add the y coordinate. I'm not sure how
- 25 Liz: So what does that bit represent? (*pointing to $-mp + q$*)
- 26 Paul: This bit here?
- 27 Liz: Hmm.
- 28 Paul: This represents the point where it crosses the origin, is that right? The c , the c part
- 29 Liz: h m m
- 30 Paul: so that bit will equal c when $p = 0$, so it's going to cross the y -axis, it's going to cross the y - oh..... because $mp + q$ except it's passing through a point, isn't it, so it's not going to be (*inaudible*)
- 31 Liz: What are you doing?
- 32 Paul: It's, I was assuming that p was the x -axis
- 33 Liz: Uhmhm
- 34 Paul: as opposed to a point.
- 35 Liz: Right.
- 36 Paul: An x coordinate
- 37 Liz: yes
- 38 Paul: whereas actually it would be where $x = 0$ that that equals c . So that bit there (*pointing to $-mp + q$*), is the point where it crosses the y -axis, but I'm not quite sure why, why that should be - perhaps it just is
- 39 Liz: Can you do me a sketch of $y = mx + c$.
- 40 Paul: $y = mx + c$. It's just going to be a line
- 41 Liz: Hmm
- 42 Paul: Could be any line
- 43 Liz: Well, umm, okay select an m and c and do it.
- 44 Paul: This is the cheat's way out - where m was 1 and c was 0. (*draws a sketch of $y = x$*)

- 45 Liz: Okay.
- 46 Paul: $m = 1, c = 0$ (reading this out as he writes it alongside his sketch) If I just keep, if I change, if I change this c value, it just, and keep the gradient the same and just move the line up and down, the axis so $m = 1$ and $c = 1$, just move up like that across the point. (draws a sketch of $y = x + 1$) Whereas if I change the gradient from the original one it - it's just basically going to get steeper (draws $y = 2x$ and writes beside it $m = 2, c = 0$) or shallower like that (draws $y = \frac{1}{2}x$ and writes $m = .5, c = 0$)
- 47 Liz: Okay. Umm, have another piece of paper and umm draw a couple of axes and put on them point (p, q) wherever you like (Paul draws a diagram with a point (p, q) marked in the positive quadrant) Umm, let's select a value for the gradient, say 1 cos that's an easy one.
- 48 Paul: so it's going to go about ... there (he draws a line through (p, q) with gradient approximately 1)
- 49 Liz: ... Umm, what's the point where it crosses the y -axis?
- 50 Paul: It's going to be.. $q - p$ is it? .. one down for every one .. Yes, it's going to cross at $q - p$ if you prove this is 1.
- 51 Liz: Yes. What if the gradient were 2?
- 52 Paul: Gradient 2, that would cross at $q -$, ... umm, ... cos it would be steeper so it would cross the y -axis down here somewhere .. So it would be, .. no, because for every one across it would be 2 down, so it would be $q - 2p$.
- 53 Liz: Right.
- 54 Paul: Yes, so it's, it's that point minus the gradient times the other coordinate, which is what we've got here (referring back to the first sheet), almost. We've got the gradient times, we've got the gradient times the first point
- 55 Liz: yes
- 56 Paul: so we've got to add, we've got to add the y coordinate so maybe if we changed, changed that round or something (writes $y = mx + mp - q$). Is that right? Can I do that, change the
- 57 Liz: [laughter]
- 58 Paul: I didn't think you could
- 59 Liz: No it's a bit naughty really. If you just write them the other way round though, and it's $+q$ and it's $-mp$

- 60 Paul: h m m
- 61 Liz: we could write them as $y = mx + q - mp$.
- 62 Paul: (*writes $y = mx + q - mp$*)Yes. Yes, umm, that right because you can see that's where it crosses the
- 63 Liz: Okay, umm, do you know any other formulas connected with finding equations of straight lines?
- 64 Paul: Umm, not apart from that one and the
- 65 Liz: hmm, $y =$
- 66 Paul: $y = m$
- 67 Liz: $mx + c$.
- 68 Paul: There's the, .. I had one for the tangent, oh no you can write it in a form like this can't you. $ax - by + c = 0$. (*writes this equation*).... You can do it like that can't you?
- 69 Liz: Yes. It should be plus (*pointing to $-by$*)
- 70 Paul: Oh yes
- 71 Liz: Okay, try erh a slight variation on that question then. Umm could you find the equation of the line which passes through the point (p, q) , and the point (r, s) ?
- 72 Paul: It's sort of umm .. yes .. it would be .. cos you've got .. (*writes a he speaks*) (r, s) (p, q) so the gradient is going to be .. $p - r$ then umm it would be $q -$ (*writes $\frac{p-r}{q-s}$*) change in y over change in x . so it's the other way round isn't it. Change in y over change in x . So that's, that's going to be the gradient and so you'd then have $y = (\frac{q-s}{p-r})x$ + wherever it crosses the y -axis which would presumably - ah it's going to be different isn't it? Oh no it's not we've got the gradient .. add q (*writes $+q$*) add - do it in terms of r and s this time, ... add - ah, no, I've just realised that we times the gradient by the - you add the y + s (*replaces $+q$ by $+s$*) $-(\frac{q-s}{p-r})r$, ... so if I rearrange it in terms of $ax + by$ (*writes $(\frac{q-s}{p-r})x - y + s - (\frac{q-s}{p-r})r = 0$*) that's the equation
- 73 Liz: It's pretty horrible isn't it?
- 74 Paul: It's really horrible. Umm there's not much you can do erh.
- 75 Liz: No there's not much you can do. I mean if I had that I think I would multiply everything through by $p - r$

- 76 Paul: yes
- 77 Liz: get rid of the fractions, but umm, after that
- 78 Paul: that, then it would be longer anyway because you get $(p - r)$ s on everything else
- 79 Liz: That's right, yes. Yes. It wouldn't improve it a great deal. Okay, I think we'll leave that one. That's marvellous. Umm, do either four or six.
- 80 Paul: I'll do six. (*Question six reads 'Find the equation of the tangent to the curve $y = x^2 + 1$ which passes through the origin'*)
- 81 Liz: Okay.
- 82 Paul: Umm presumably it means that that's umm, that's that curve that goes through the tangent, goes through the origin not the tangent.
- 83 Liz: No it's the tangent that goes through the origin.
- 84 Paul: The tangent goes through the origin, oh
- 85 Liz: Because that curve doesn't go through the origin does it?
- 86 Paul: No it goes through there (*he draws two axes and indicates the point $(0, 1)$*). go through ... It wouldn't come in the minus, it wouldn't cross the x-axis, because that value (*referring to the $+1$*) would have to be minus to cross the x-axis
- 87 Liz: yes
- 88 Paul: so it doesn't it goes like that (*he draws a straight line of approximate gradient 1 going from $(0, 1)$ into the positive quadrant*) ... at a gradient of x no, it just goes like that.
- 89 Liz: What sort of curve is this?
- 90 Paul: Oh silly me, it's going to be like that (*he draws the right hand half of a parabola with a minimum vertex at $(0, 1)$*),
- 91 Liz: Uhmhm
- 92 Paul: then it's going to go like that at the same, in the same way (*he completes the left-hand half of the parabola*)
- 93 Liz: right
- 94 Paul: it's a parabola.
- 95 Liz: Yes.

- 96 Paul: Now I'm . somewhere along here there's going to be a tangent to the curve that goes through the origin, ... so there's two possible ones it could be.
- 97 Liz: Hmm.
- 98 Paul: It could be one there, and it could be one there. *(he draws both possible tangents on his diagram)*
- 99 Liz: Okay. Do you want to decide which one you're going to find and then we can come back to the second
- 100 Paul: Stick to the positive one first
- 101 Liz: Well that's a surprising choice! *[laughter]*
- 102 Paul: Ah, so now I'll try, because I know that umm the equation for a - no that's a normal - tangent - tangent going to have the same .. I've forgotten the equation for a tangent.
- 103 Liz: I don't think you know a formula for it.
- 104 Paul: No, I just know the way to find the gradient.
- 105 Liz: Hmm. Is this question harder than it looked?
- 106 Paul: Hmm. Yes.
- 107 Liz: Can you put your finger on why it's difficult?
- 108 Paul: Because you've, you've got no place to start after you've done that. You know that it passes through the origin, it could do that, it could do that, whatever, be a tangent to the curve and pass through it. I don't know, you can only do it one. Well you know it passes through the origin but you've got no, no idea where it touches
- 109 Liz: right
- 110 Paul: and you need to know where it touches to be able to get the gradient
- 111 Liz: Uhmhm
- 112 Paul: and you need to have the gradient to know where it touches.
- 113 Liz: Yes.
- 114 Paul: So you've got a loop which you can't, you can't solve that easily.
- 115 Liz: Right
- 116 Paul: No it's hard

- 117 Liz: Let me give you a general strategy. Working on a problem like that, I can't get anywhere without knowing something, a piece of information and I've got no way of finding it out.
- 118 Paul: Uhmhm.
- 119 Liz: One strategy which will work in most cases, quite often it does work, umm is to pretend that you do know it
- 120 Paul: and then work and see if it works.
- 121 Liz: Yes.
- 122 Paul: up , say I've taken the gradient as 1, then the gradient function of that is going to be a point on the curve where the gradient is also 1, isn't it?
- 123 Liz: Uhmhm.
- 124 Paul: And the point that it's going to be 1 isn't as it is, no, yes it's going to be at 0.5 when x is 0.5.
- 125 Liz: Uhmhm.
- 126 Paul: So I'm going to put 0.5 erh at a gradient of 1, it's not going to go through the origin. Oh no, x - that's got a gradient of 1 passing through the origin, then the equation is $y = x$, got that one, yes they're going to be different. There's going to be a different value for that curve than that. That one sort of can't be that. For a gradient of 2, x would be 1, which would give you 2, $y = 2x$, ... that would be - yes that's wrong as well.
- 127 Liz: Could you describe the checking process that you're going through.
- 128 Paul: Oh yes, sure.
- 129 Liz: I think I know what you're doing.
- 130 Paul: What I'm doing is that I'm taking erh I'm giving the line a gradient
- 131 Liz: yes
- 132 Paul: like say I'm going to do it with 2, then I'm using, I'm differentiating that to find where the gradient would equal to it,
- 133 Liz: right
- 134 Paul: which would be the x - point - .
- 135 Liz: Uhmhm.

- 136 Paul: 1 in this case, then I'm putting the 1 into that (*indicating the curve $y = x^2 + 1$*), so y would equal 2.
- 137 Liz: Right.
- 138 Paul: Then I'm going back to my equation on the line with the gradient 2, passing through the origin and seeing if that value I get for y there is the same as that value I get for y and putting it through the equation of that , to see if they match.
- 139 Liz: Right.
- 140 Paul: 1 times 2 .. $y = 2x$.. it does, it does match that. .. $y = 2x$ so y would be 2, x was 1, $x^2 + 1$, $y = 2$ Yes that's right.
- 141 Liz: Right. So that point's on both these lines.
- 143 Paul: Yes, and it passes through the origin
- 144 Liz: and the gradients match.
- 145 Paul: Yes so the gradients to the equation of, if I answer to the question, the equation is going to be $y = 2x$, - no x is going to, we've got a gradient which is .. $y = 2x + 0$. Yes, I think that's right.
- 146 Liz: Okay. That umm sort of trial and error
- 147 Paul: Uhmhm
- 148 Liz: method worked pretty well on that.
- 149 Paul: Uhmhm.
- 150 Liz: Umm because it was quite a simple answer.
- 151 Paul: Because, now if I were to work out the other one, I could take a normal to that one passing through (0, 0), because it would be perpendicular I think. Normals are perpendicular aren't they?
- 152 Liz: yes normals are perpendicular
- 153 Paul: you, and so, if you had, and this curve is symmetrical no it isn't, yes it is, yes it's symmetrical, so surely if this one goes that way, then it's going to match one coming that way.
- 154 Liz: Yes but not necessarily at right angles - it could be like that, or like that, or like that.
- 155 Paul: Yes, you were going on about trial and error - worked in that case.....
- 156 Liz: Just erh finish off what you were doing since (*inaudible*) and we'll come back to that

- 157 Paul: -1 2, yes I think it is. Because if you have one with a gradient of -2, which you should have, then that's going to equal -2, $x = -1$, and $y =$, well $x = -1$, $y = 2$, on this one, and on the equation $y = -2x$, it equals 2 as well. So it's, it's equation would be (*writes* $y = -2x + 0$). Or just $-2x$.
- 158 Liz: Okay, right. So we've got them both. Umm so yes, I was saying it's, it's worked quite well here because it was a fairly simple equation to find,
- 159 Paul: Uhmhm
- 160 Liz: but umm you can imagine that if the answer was $y = 3/7$ th x , it might take a bit longer. Umm so just to give you an idea about refining your method, the basic idea is absolutely fine, you've picked on something that you were going to fix and then you checked it out.
- 161 Paul: Uhmhm.
- 162 Liz: Well, try this instead. You've picked a gradient, first you tried 1, then you tried 2. Instead of doing that, let's try m .
- 163 Paul: m ?
- 164 Liz: Hmm, for the gradient.
- 165 Paul: Okay.
- 166 Liz: Erh and work through the same thing as you did but using m , and in the process we should be able to find out something about m .
- 167 Paul: So write an equation where I can put m in.
- 168 Liz: Well .. umm ...yes, think back to what you did to start with. You said 'suppose the gradient's 1,'
- 169 Paul: oh, yes.
- 170 Liz: Now go back to that stage and think 'suppose the gradient's m .'
- 171 Paul: If you put $y = mx + c$ (*inaudible*). You know it's zero, you know it's $y = mx$
- 172 Liz: yes
- 173 Paul: equals, ah $m = 2x$ doesn't it, because the gradient's - m is going to equal $2x$, so it's $y = 2x^2$
- 174 Liz: .. hmm
- 175 Paul: because that's $2x$ times

- 176 Liz: yes
- 177 Paul: x , is that going to work?
- 178 Liz: What do you think?
- 179 Paul: I, I don't see why it shouldn't do, because you've, you've got - oh no it's going to be $2x$ x isn't it?
- 180 Liz: h m m
- 181 Paul: as opposed to $2x^2$ because you're going to want the $2x$ times the x .
- 182 Liz: Uhmhm.
- 183 Paul: .. But that's what your gradient is.
- 184 Liz: Yes.
- 185 Paul: It is, yes. and if we could put that into the form, if we put that into having a y , could we do simultaneous equations to try and find out where they're the same, so you had That, is that $2x$ times x different from $2x^2$? It is isn't it?
- 186 Liz: Well yes and no. Umm
- 187 Paul: As it's minus - no it isn't
- 188 Liz: What's giving you the suspicion that they're different sorts of x s?
- 189 Paul: .. It's because one has been differentiated, one hasn't
- 190 Liz: h m m
- 191 Paul: so you have an x^2 there, maybe it's been distorted somehow or other.
- 192 Liz: ... You said a minute ago that $m = 2x$.
- 193 Paul: Uhmhm. Taking it from the fact that the gradient of this one is $2x$, so if they're going to match, that value of m there has got to equal the $2x$.
- 194 Liz: Umm the $2x$ that you're thinking about there
- 195 Paul: Uhmhm
- 196 Liz: is actually the value of $2x$ at the right point on the curve. So that $2x$ actually takes a particular value that you don't know yet.
- 197 Paul: So this is going to be mx , mx equals, so that's (*writes*
 $mx - y = x^2 - y + 1$)
- 198 Liz: Uhmhm

- 199 Paul: Is that right? you can add y to both sides
- 200 Liz: yes
- 201 Paul: which would give us $mx = x^2 + 1$ (*writes* $mx - 1 = x^2$, $m - \frac{1}{x} = x$)
..... I want x to equal ..
- 202 Liz: hmm, what are you trying to do with that equation?
- 203 Paul: I was trying to get it down to a value of x , or x equalled but maybe I should be trying to get it to what m equals.
- 204 Liz: Yes.
- 205 Paul: (*writes* $m = x + \frac{1}{x}$) Yes, because $m = x + \frac{1}{x}$ m
can change can't it?
- 206 Liz: What do you mean by m can change?
- 207 Paul: Well the gradient can change, it's always through the same point, and that the gradient of a line can change
- 208 Liz: h m m
- 209 Paul: so, say you had the point (2, 2), you could have a gradient of 1, you could have a gradient of 4, you could have a gradient of anything and if you had $1 = 2 + 1$ it doesn't go
- 210 Liz: Uhmhm
- 211 Paul: It's only for that, only for that particular equation. For those two to match m has got to equal that.
- 212 Liz: Right. Umm ... the x that you've got in that equation there, let's take one step. How did you form that equation? (*referring to* $mx - y = x^2 - y + 1$)
- 213 Paul: This one here?
- 214 Liz: Hmm.
- 215 Paul: I took the equation of both lines
- 216 Liz: right
- 217 Paul: put them together so they've got to equal each other
- 218 Liz: h m m.
- 219 Paul: Oh no that's wrong, isn't it. That's a, to find a point on a line that equals each other, when those two lines touch,

- 220 Liz: h m m
- 221 Paul: that's what that's up to finding
- 222 Liz: yes. So that's not necessarily wrong because we are looking for the point where the line touches the curve, aren't we?
- 223 Paul: Uhmhm. I suppose so
- 224 Liz: So that equation that you ended up with is true at the point where the line and the curve touch.
- 225 Paul: That point yes. We found these two touched at 2 and, 2 and 1. That worked for them all. (*inaudible*) - yes it does.
- 226 Liz: Right.
- 227 Paul: So that is true.
- 228 Liz: But at the moment we've only got that one equation, and we don't know what either x or m is, well in fact you do because you've already done the question, but you wouldn't do
- 229 Paul: no
- 230 Liz: umm so one equation is not enough.
- 231 Paul: .. If I took .. away the one with the y , - no, that's not going to help. ... Basically we can't do much till we find the value of one of, one of the
- 232 Liz: hmm yes, well you've got one equation linking m and x .
- 233 Paul: Uhmhm.
- 234 Liz: If you got another equation linking m and x then we could solve the pair of them.
- 235 Paul: Yes, so I could go, ... with just m and x in them?
- 236 Liz: Well we don't want anything else that we don't know.
- 237 Paul: Uhmhm. We've got an $m =$, we've got an $x =$.
- 238 Liz: Right.
- 239 Paul: Do you mean find another equation for, linking those two together yes?
- 240 Liz: Yes.
- 241 Paul: But not in that form
- 242 Liz: Uhmhm

- 243 Paul: so that we could then use the information contained in them to work out the two values. We could do that by a simultaneous equation.
- 244 Liz: Yes.
- 245 Paul: So what, why can't we just do, (*writes $m - \frac{1}{x} - x = x + \frac{1}{x} - m$*) or isn't that enough?
- 246 Liz: umm that boils down to nothing equals nothing, doesn't it?
- 247 Paul: Okay. Yes, it does.
- 248 Liz: No it doesn't actually. It boils down to the same as the one above it.
- 249 Paul: Uhmhm. It boils down to $\frac{1}{x} - x = x + \frac{1}{x}$
- 250 Liz: Hmm.
- 251 Paul: Because you've taken to just
- 252 Liz: Umm in forming that equation, the information you've used is just that at this point that we're interested in, the line touches the curve. Now think back to what else you were checking out when you did it by your trial and error method.
- 253 Paul: .. I was like checking out to see if it worked.
- 254 Liz: Hmm.
- 255 Paul: I was just checking out that, that the two points, the two y values match for that one and the equation for that line.
- 256 Liz: Yes there was umm there was a first stage though.
- 257 Paul: What finding the gradient via that one, finding the point on the curve
- 258 Liz: yes
- 259 Paul: where it had the same gradient as the line
- 260 Liz: yes
- 261 Paul: .. the values are like when $m, m =$ then x will equal, something like that
- 262 Liz: Yes.

- 263 Paul: So how's that going to help me find two linking equations?
- 264 Liz: Well the thing is that umm at the moment we don't know which point on the curve we're talking about.
- 265 Paul: Oh right.
- 266 Liz: Umm but when you did it by trial and error you said, this was the first stage I think, the gradient's 1
- 267 Paul: Uhmhm
- 268 Liz: now the point on the curve at which the gradient is 1
- 269 Paul: yes
- 270 Liz: is where $x = \frac{1}{2}$.
- 271 Paul: Uhmhm.
- 272 Liz: So the point that I'm checking out is where $x = \frac{1}{2}$. Then I check
- 273 Paul: ah it's so we can put m for the value that we want, we can give that a value of m .
- 274 Liz: Hmm.
- 275 Paul: We could do it in terms of the point at which they are the same.
- 276 Liz: Yes.
- 277 Paul: Ah right. .. So, is there somehow we can put $2x$, no, no.
- 278 Liz: You know what you just said that I said yes to
- 279 Paul: hmm, what
- 280 Liz: I didn't really understand it. I thought I meant yes, but I'm not sure now. Can you tell me again?
- 281 Paul: Which bit?
- 282 Liz: Umm
- 283 Paul: Oh, I said well if we umm if we had the point where they touched
- 284 Liz: Uhmhm
- 285 Paul: but at, by giving that point a value of $\frac{1}{2}x$, no, no that's wrong, umm no, well yes, the gradient equals, the gradient equals $\frac{1}{2}x$,

- 286 Liz: $2x$
- 287 Paul: Yes, well no, $\frac{1}{2}$ of, I was thinking more of $\frac{1}{2}x$ because you've got to times it by 2 to get the gradient. oh, except, yeah it is $2x$
- 288 Liz: Erh no, you've got to halve something to get something.
- 289 Paul: Well, you've got to times something by, your gradient is twice x .
- 290 Liz: Right. So your x is half the gradient.
- 291 Paul: Yes, (*inaudible*)
- 292 Liz: That's actually a more useful way of thinking of it.
- 293 Paul: ... Yes because then you haven't got to do anything to solve x .
- 294 Liz: [*laughter*]. Yes.
- 295 Paul: So couldn't we just have that $x = \frac{1}{2}m$
- 296 Liz: Hmm.
- 297 Paul: Then put it together with one of these.
- 298 Liz: Yes. Marvellous
- 299 Paul: So we've got $\frac{m}{2} - x = \dots$ which one do we want - we want the one with the $-x$.
- 300 Liz: Which one's that?
- 301 Paul: Would we want this one where we put $m - x$, (*pointing to* $m - \frac{1}{x} - x$, *which is the left hand side of an equation he has written down earlier*) or if you like, because then it'd just be easier to negate it down,
- 302 Liz: Uhmhm
- 303 Paul: if you've got sort of $-xs$ on both sides
- 304 Liz: h m m
- 305 Paul: you can get rid of them quicker.
- 306 Liz: I can't see what's happening here. Which equation are you substituting them into?
- 307 Paul: That's this one. $x = \frac{m}{2}$ so that (*pointing to* $\frac{m}{2}$) minus x equals 0.

- 308 Liz: Right. Yes, don't carry on Paul.
- 309 Paul: Uhmhm.
- 310 Liz: Umm this is a very strange way of solving simultaneous equations.
- 311 Paul: Oh I always put whatever equals 0.
- 312 Liz: Yes, the problem with that is you've still got x s and m s then. The erh, the key aim in solving the pair of equations is to get rid of one of the variables so you can find out what the other one is.
- 313 Paul: Uhmhm.
- 314 Liz: So umm you want to use this equation to substitute in to
- 315 Paul: oh right then
- 316 Liz: one of your others.
- 317 Paul: So I'd have,
- 318 Liz: Do you want some more paper?
- 319 Paul: Yes please - in that case, so basically I'd have this equation here (*writes* $m - \frac{1}{x} - x = 0$). I'd have $m - \frac{1}{x} - \frac{m}{2} = 0$. Now I've still got m s and x s
- 320 Liz: Uhmhm
- 321 Paul: so I'd do ... $2x - \frac{1}{x} - x = 0$, which comes down to $x - \dots\dots\dots$ (*writes* $x - \frac{1}{x} = 0, x = 1$)
- 322 Liz: Uhmhm.
- 323 Paul: Yes. Which is, yes, and then we've got, because $x = 1$, it will fit back into our equation and we know that the gradient is therefore, and then we can put them back into those, and we can solve them.
- 324 Liz: Right. It's easier by trial and error isn't it?
- 325 Paul: Yes (*laughter*). Oh, I don't know actually, it's that if you didn't have ones - it's good to be able to know you could do it.
- 326 Liz: Hmm.
- 327 Paul: That way's to check it up.

- 328 Liz: Yes. So, take a minute to think about it and then tell me what's going on in this question, what's it about?
- 329 Paul: It's about simultaneous equations in the end. Cos it's about, it's about interpreting your information and taking it to something else.
- 330 Liz: Uhmhm.
- 331 Paul: As in you could, you could do it, you could tackle it in a number of different ways. You could use the trial and error way using the graph and putting things into the two equations, or you could try and take bits out and rearrange it there. So it's, it's not so much a graphical question
- 332 Liz: h m m
- 333 Paul: as an algebraic one.
- 334 Liz: Hmm, right. Umm
- 335 Paul: They, they could quite easily have said when does $y =$, when does $y =$ that = that. When do they have, they could have phrased it a lot differently without having mentioned the graph.
- 336 Liz: Hmm.
- 337 Paul: And they're still asking you the same question, maybe giving you a bit more umm information.
- 338 Liz: Yes. The, you didn't know the gradient, and that was really what they were asking you to find out, the equation of the tangent cos if you know the gradient that will give you the equation
- 339 Paul: Uhmhm
- 340 Liz: umm and you didn't know the point where the tangent was a tangent to the curve.
- 341 Paul: Uhmhm.
- 342 Liz: Umm, and the way you chose to do it was to guess the gradient and check it out, umm and I've shown you a way of supposing you know the gradient to be m and then doing the checking out process
- 343 Paul: Uhmhm
- 344 Liz: and finding out what m is by doing that.
- 345 Paul: Hmm.

- 346 Liz: Umm, another way of approaching it would have been to say 'well I don't know what the point is where they touch, but suppose I do, let's call it
- 347 Paul: yes
- 348 Liz: something or other', and then do some checking out on that to find out what the point is.
- 349 Paul: You do that, like say call the point P, say you've called the point P
- 350 Liz: yes, or you'd have two coordinates
- 351 Paul: (*inaudible*), now but how would you do that if you'd got two variables, because it would equal, would we know this at the time that it would equal umm $\frac{1}{2}x$ umm (*writes* $(p, q) = (\frac{x}{2}, ..)$)
- 352 Liz: half what?
- 353 Paul: $\frac{1}{2}x$, you know, because with the x coordinate it's got to be sort of thinking about in terms of the gradient.
- 354 Liz: Yes, the x -coordinate had to be half of the gradient didn't it?
- 355 Paul: Yes. It would have to be $\frac{m}{2}$, (*writes* $(p, q) = (\frac{m}{2},)$ m over, ... you would then have to substitute into all of that
(*completes the bracket with* $m \times \frac{m}{2}$) You'd have to have that
- 356 Liz: Uhmhm.
- 357 Paul: That's mx
- 358 Liz: yes
- 359 Paul: $y = mx$.
- 360 Liz: Yes. .. So I can see there that you've used two pieces of information.
- 361 Paul: Yes, I've used the information we've got that $x =$ the gradient over the
- 362 Liz: gradient over two.....
- 363 Paul: equals half, yes, half a gradient.
- 364 Liz: Where did that piece of information come from?
- 365 Paul: Erh it came back from the bit we already knew

- 366 Liz: yes
- 367 Paul: but we don't know if that bit's okay
- 368 Liz: Well it's okay to use that.
- 369 Paul: Uhmhm.
- 370 Liz: Can you remember why you already knew it? Where did it come from originally?
- 371 Paul: It came from the fact that the gradient function, where they touched m had to equal half of x because it equalled $2x$, the gradient equals $2x$.
- 372 Liz: Yes.
- 373 Paul: So x equals half of that.
- 374 Liz: Okay, I'm sorry, that was a distraction. You were saying we've used the fact that x is half the gradient.
- 375 Paul: Uhmhm, the gradient is half of x .
- 376 Liz: Yes. Uhmhm.
- 377 Paul: So but x is half of
- 378 Liz: [laughter]
- 379 Paul: and we just put, we've just substituted that into a $y =$ on the tangent one because it's simpler, because you get gradient times $x = y$. So you've got the gradient times gradient over two.
- 380 Liz: Right. So you've used the fact that the gradients match. You're effectively saying the gradient of the tangent is the same as the gradient of the curve
- 381 Paul: Uhmhm
- 382 Liz: by putting in $\frac{m}{2}$
- 383 Paul: yes
- 384 Liz: and you're also saying this point lies on the tangent because the y coordinate is m times the x coordinate.
- 385 Paul: Uhmhm.
- 386 Liz: Also that point lies on the curve.

- 387 Paul: Uhmhm, yes. It has to lie on both. So that substitute that m in both, both of the equations, so we could have $y = \dots\dots\dots$ (writes $y = (\frac{m}{2})^2 + 1, y = \frac{m^2}{2}, \frac{m^2}{2} = (\frac{m}{2})^2 + 1$)
- 388 Liz: Uhmhm.
- 389 Paul: Then use, $\dots\dots\dots$ if we call that (pointing to the 1 in the equation) $\frac{m}{m}$ or wouldn't that work?
- 390 Liz: No, I don't think that would help.
- 391 Paul: No.
- 392 Liz: Umm try multiplying out that $(\frac{m}{2})^2$.
- 393 Paul: $\dots\dots\dots$ (writes $\frac{m^2}{2} = \frac{m}{2} \times \frac{m}{2} + 1$) Is that right?
- 394 Liz: Uhmhm.
- 395 Paul: ... Keep the balance. Off here keep that as $\frac{m}{2}$ and (writes $m \frac{m}{2} = \frac{m}{2} \times \frac{m}{2} + 1$)
- 396 Liz: $\dots\dots\dots$ umm, get rid of the fractions?
- 397 Paul: $\dots\dots$ Is this (pointing to $m \frac{m}{2}$) going to become $2m \ m$?
- 398 Liz: I would carry on calling it $2m^2$ myself.
- 399 Paul: Yes. $\dots\dots\dots$ (writes $2m^2 = m \times m + 2$)
- 400 Liz: There's something funny going on here. What did you multiply through by to get rid of the fractions?
- 401 Paul: 2.
- 402 Liz: Hmm, yes. .. You must have multiplied by 2 twice to get rid of both of those (pointing to the two 2's in $\frac{m}{2} \times \frac{m}{2}$). And in fact you did it by 2 twice here (pointing to $m \frac{m}{2}$) because you cancelled that one out and you also put another one in.
- 403 Paul: So it's, (changes the + 2 to + 4)
- 404 Liz: yes.

405 Paul: Carry it across That's , so we get ... we can't divide through by the .. and we get

406 Liz: Hmm, try working out what m^2 is.

407 Paul: I, I know what it is. I can see it. I was just trying to think of like u m m

408 Liz: You've forgotten how to do these.

409 Paul: No that's completely wrong. It should be ... that's right isn't it?

410 Liz: No.

411 Paul: Move the, the four to that side. So we've got $2m^2 - 4 = m^2$

412 Liz: Yes, but why? You've got two terms in m^2 .

413 Paul: .. Oh I wonder ifyes $3m^2 = 4$.

414 Liz: ... $2m^2$ here

415 Paul: Uhmhm

416 Liz: but this m^2 is on the other side.

417 Paul: Uhmhm.

418 Liz: When you bring them together,

419 Paul: .. minus it.

420 Liz: Hmm.

421 Paul: So it's, it would be just be $m^2 = 4$.

422 Liz: Right.

423 Paul: That would be $m = 2$.

424 Liz: Hmm. Or -2 in fact.

425 Paul: Yes. because (*writes $m = \pm\sqrt{4}$*)

426 Liz: Yes. So that gives you both answers.

427 Paul: Uhmhm.

428 Liz: Yes. Okay.

429 Paul: Uhmhm.

430 Liz: Right. Well I'm not going to ask you to do another one because it's getting on for lunch time

- 1 Liz: They're reasonably straight forward I think. What I'd like you to do is
- 2 Robert: ah hmm
- 3 Liz: work through them, tell me what you're doing, umm and then I'm probably going to ask you some more probing questions about what it is you've done, umm, so could we start with that one (*I give him question 4 which reads 'Find, in terms of a and b , the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$*). Have a pencil.
- 4 Robert: Ah, hmm, this is a vectors problem, I'm thinking.
- 5 Liz: Ah ha.
- 6 Robert: Ah umm what I'll do
- 7 Liz: Do you know what it is that strikes you about it that says vectors?
- 8 Robert: Umm well this point (a, b) and the word perpendicular and umm
- 9 Liz: right
- 10 Robert: recent 'A' level questions.
- 11 Liz: Ah ha.
- 12 Robert: Which said this very thing. So I'd umm think I'd get a - what I'm going to do is find erh the distance from the point a to the general point on the line, but, - the point (a, b) to the general point on the line,
- 13 Liz: ah hmm
- 14 Robert: and then I'm going to minimise it.
- 15 Liz: Right.
- 16 Robert: Because the perpendicular's got the minimum distance
- 17 Liz: yes
- 18 Robert: Umm, you want me to write on here?
- 19 Liz: Ah hmm.
- 20 Robert: I can't remember what form I want this line, this is what I'm trying to think. Hmm. I want it, I want it like in vector form, .. I can't think how I put it in there, which is annoying.

Well I'll have y is equal to, so y is equal to $2 - \frac{1}{2}x$ I'll have a drawing too. *(he draws a diagram showing a line going through $(0, 2)$ and with gradient $\frac{1}{2}$)* I have a point (a, b) , *(he marks a point (a, b) in the positive quadrant below the line)* hmm, I want this, *(he draws the perpendicular from the point to the line)* .. distance here and then umm, hmm, I was just thinking there about seeing this right angle, I was thinking that if I could find a point here, I could use it, dot two things to find this length, but I don't think that's possible really. Erh .. and now I could make, make up my vector equation of this line because I can

21 Liz: okay

22 Robert: see more easily what it is. *(he writes $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}$)* It's two across and one up.

23 Liz: Ah hmm.

24 Robert: So I think that's right. .. So that would say that it goes through, I'm going to just check this, goes through the point $(2, 3)$, erh which is right, umm

25 Liz: yes, I should just stop you there because I've just realised that your diagram was wrong. Umm

26 Robert: Is it?

27 Liz: It's gradient of negative half, umm so that's sloping the wrong way.

28 Robert: Ah yes. That's silly, doesn't change it too much though. I mean to say this is, this is positive x here, this is minus x . *(he alters his diagram)* Umm does make it harder to think of though, so you just put -1 in here. *(he changes the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$)*

29 Liz: Right.

30 Robert: Umm and now we have to find the, I'm using this as the point on a line *(he re-writes the position vector \mathbf{r} as $\begin{pmatrix} 2\lambda \\ 2 - \lambda \end{pmatrix}$)* umm and then we do the point umm (a, b) *(he writes $P = \begin{pmatrix} a \\ b \end{pmatrix}$)*, .. and we do erh, .. to go from the point on the line to P *(he writes $\mathbf{r}P =$)*, we have to go $a - 2\lambda$, and then -

31 Liz: Interesting notation.

- 32 Robert: Yes, exactly. And then we go $b - 2 + \lambda$, (*he has written* $\underline{r}P = \begin{pmatrix} a - 2\lambda \\ b - 2 + \lambda \end{pmatrix}$) and we want to minimise, minimise this vector. So the length is going to be l^2 (*writing as he speaks*), well length is l
- 33 Liz: ah hmm
- 34 Robert: and square is going to be $(a - 2\lambda)^2 + (b - 2 + \lambda)^2$, $a^2 + 4\lambda^2 - 4a\lambda + b^2 + \lambda^2 - 4\lambda + 4 + 2b\lambda - 4b$ (*writing as he speaks*), I think. That looks quite good. $a^2 + b^2$
- 35 Liz: Looks pretty horrible to me!
- 36 Robert: Umm this is what you normally get when you do these
- 37 Liz: ah ha, yes
- 38 Robert: something along these lines, and you can't really do much. That's $5\lambda^2$ (*he continues simplifying the previous expression*). There is a formula for these lines and points but I don't remember what it is. Not really on the syllabus. You've just got to do the best with what you've got left. λ into (*he now has written* $a^2 + b^2 + 5\lambda^2 - \lambda(4 + 4a - 2b) - 4b$) I mean you .. then you've just got to find the square root of this and that's the length l
- 39 Liz: ah hmm
- 40 Robert: Ah yes we're looking at completing the square, that was the one. λ .
- 41 Liz: Yuk!
- 42 Robert: Yes, it's getting a bit nasty
- 43 Liz: Why don't you differentiate it instead?
- 44 Robert: Yes, that'll be easier, but I think I'll leave it as l^2 .
- 45 Liz: Yes.
- 46 Robert: Umm, .. through differentiating it with respect to λ It's going to be $10\lambda - 4 + 4a$. .. Hmm, $-4a + 2b$ (*he has written* $\frac{dl^2}{d\lambda} = 10\lambda - 4 - 4a + 2b$), then (*inaudible*) if you've done all this I think, equals 0, then you put that back in and you get a value.
- 47 Liz: Ah hmm.
- 48 Robert: I don't really feel like doing that now.

- 49 Liz: Okay.
- 50 Robert: It's a lot of work.
- 51 Liz: Right. Umm what, so what sort of form are you expecting for the answer then if you put it back in and so on.
- 52 Robert: Umm .. because I've, I've certainly looked at this before, in a book
- 53 Liz: ah hmm
- 54 Robert: and I'm expecting something like $(a^2 + b^2)$ over the square root of something or I don't really know, it would be a fraction though, of the length I think
- 55 Liz: all right
- 56 Robert: I don't know. But you, but it should, $-4ab$ or something like that, don't know but it, it's going to be quite small I think.
- 57 Liz: Are you erh, are you, you're recalling a formula which is something like umm erh $\frac{ag + bh + c}{\sqrt{(a^2 + b^2)}}$.
- 58 Robert: Yes, that sounds good.
- 59 Liz: Yes. Can you remember what that formula's for?
- 60 Robert: Erh finds the distance from a point to a line.
- 61 Liz: Right. Umm, now it has an a , b , c and a g and a h in it, the version that I remember.
- 62 Robert: Yes, erh, I don't remember what they are though.
- 63 Liz: Well the a , b and c refer to the equation of the line
- 64 Robert: yes
- 65 Liz: umm like the equation of the line is taken to be $ax + by + c = 0$.
- 66 Robert: Yes, that will be it.
- 67 Liz: The g and the h are the coordinates of the point.
- 68 Robert: Yes I don't know this. This is getting pretty horrible but I mean don't really see a great way for me doing this straight from the Cartesian, but this is a sort of question which I have, you see on an 'A' level vectors question.
- 69 Liz: Right, well I, I think they'd be pretty unkind to give you an equation

- 70 Robert: that one yes
- 71 Liz: in Cartesian form.
- 72 Robert: Umm well you're supposed to be able to get it from there to there
- 73 Liz: yes, yes
- 74 Robert: but it's not, not that difficult.
- 75 Liz: Right, umm
- 76 Robert: I might
- 77 Liz: The answer is certainly going to contain a s and b s
- 78 Robert: yes
- 79 Liz: although not squared I don't think
- 80 Robert: no.
- 81 Liz: Right, why?
- 82 Robert: Why is it going to contain a and b ?
- 83 Liz: Yes.
- 84 Robert: ... Well, could be cos it depends on where the point is, I mean obviously the point is a long way away. You need to know where the point is. You'll never find the distance from the point to a line if you don't know where the point is.
- 85 Liz: Right. Could you argue for why it's going to contain a and b in linear form rather than quadratic?
- 86 Robert: Umm, umm wouldn't expect to have it in quadratic form because when you square this point, it's going to go off somewhere else which isn't really useful.
- 87 Liz: Mmm.
- 88 Robert: In sort of a, in like a complex form - if you square it it's going to go off and that's not going to help, that's not going to tell you anything when you square it. Umm, . a^2 , the square root of $a^2 + b^2$ will tell you this distance up to there (*indicating the origin*)
- 89 Liz: h m m
- 90 Robert: so you might expect to see that.
- 91 Liz: Well remember what we're asking for is the
- 92 Robert: foot of the perpendicular to this point

- 93 Liz: coordinates at the foot of the perpendicular, yes, not the distance.
- 94 Robert: Yes, oh you're not looking for that distance?
- 95 Liz: No.
- 96 Robert: Oh, umm, .. umm, .. that would make it quite hard finding, normally they'd ask you for this length,
- 97 Liz: h m m
- 98 Robert: which is probably why I went for the length. I do that too much though, I don't read the bloody question.
- 99 Liz: [laughter]
- 100 Robert: It's really annoying too. Umm
- 101 Liz: Okay forget that one for the moment.
- 102 Robert: Yes.
- 103 Liz: Umm I think you have done something very like this question before. Umm that one. *(I give him question 6 which reads 'Find the equation of the tangent to the curve $y = x^2 + 1$ which passes through the origin')*
- 104 Robert: This one. Ah.
- 105 Liz: But I'd like you to work through it anyway.
- 106 Robert: Okay. Umm .. Oh tangents says derivatives straight away to me, $2x$ *(he writes $\frac{dy}{dx} = 2x$)* umm and so we do *(writes)* $y - y_0$ is equal to .. hmm .. I'm just going up here, it passes through the origin, I'm getting all mixed up here. .. Well it's going to be, m into $x - x_0$ Hmm what's going on here? Draw a little diagram here. We're expecting two because the .. graphs in here. We're expecting one there and one here. *(he has drawn a diagram showing the curve and two tangents through the origin)*
- 107 Liz: Ah hmm.
- 108 Robert: .. Umm *(he has written $y - b = m(x - a)$)*
- 109 Liz: What's this point? *(I am referring to the point (a, b))*
- 110 Robert: Umm this is the point which is P here. *(he marks the point of intersection of the curve and the tangent with positive gradient as P)*
- 111 Liz: Right, okay.

- 112 Robert: .. Hmm what am I doing wrong here? .. I'm recognising I'm stuck.
- 113 Liz: All right, well I know you know what to do when you're stuck.
- 114 Robert: Yes. Umm .. I've got the right gradient. Umm if I want a tangent, .. I want to .. get my line going here. And it, .. these aren't my favourite problem's with this because I can, I can never get, not very happy with this erh equation here.
- 115 Liz: Hmm.
- 116 Robert: .. They generally tend to be
- 117 Liz: What do you mean by you're not very happy with it?
- 118 Robert: Well ..I've seen this on a number of 'S'-level questions and haven't been able to do them.
- 119 Liz: Ah.
- 120 Robert: which is why I don't like them very much, and erh, well if I know that this point .. goes through the origin then surely y_0 and x_0 will be zero. Oh I don't know. I'm getting annoyed here. I feel like I can sort of start plugging values in to see what I can, to see if I can see what's going on.
- 121 Liz: Ah hmm.
- 122 Robert: So I'll guess x is 2, then you're going to get 4 as the gradient. And it's going to go through the point .. 5, .. so it's not going to come back, not going to go through the origin.
- 123 Liz: Ah hmm.
- 124 Robert: What about the point 1. Gradient of 2 goes through the point 2, so that won't go through the origin either. .. But that one will go above and the other one will go below, so it's somewhere between 1 and 2 I think. .. Umm but I'm feeling that I should be using this $x^2 + 1$ and this point (a, b) .. is x^2 - is $(x, x^2 + 1)$.
- 125 Liz: Ah hmm.
- 126 Robert: That's what I'm doing. $y - x = 2x(x - ,$ oh $x^2 + 1$ here (*he replaces $y - x$ by $y - (x^2 + 1)$*). .. $x - x$, that's good (*he has written $y - (x^2 + 1) = 2x(x - x)$*). That's really helpful. Erh .. I just can't think. I think I'm going to put t in here, .. because erh you get a lot of, a number of these questions where it's a parametric
- 127 Liz: ah hmm

128 Robert: curve. So we're $y - (t^2 - 1)$ is equal to the gradient which is $2x$ into, you've got to relate it to a point really. I don't know. Is equal to erh $(x - t)$ (he has written $y - t^2 - 1 = 2x(x - t)$). Now I've got three variables involved. And the line's not straight. Hmm Well if I'm supposed to be finding the equation of the tangent, .. I'm going to have, it's going to be of the form y is equal to $mx + c$ (writes $y = mx + c$) where c is equal to 0 (writes $c = 0$), because it has to go through the origin. So the gradient's got to be equal to mx , (writes $y = mx$) and .. this has got to intersect the curve, the equation $y = x^2 + 1$ (writes $y = x^2 + 1$). So you've got mx is equal to $x^2 + 1$ (writes $mx = x^2 + 1$). Ah Hmm. I'm not sure what this has told me by finding what x is. there's telling me the x coordinate of this point here. Hmm, at a guess I'd say that that point would be root 2. ... I'm not sure why

We are interrupted by someone who has come to call Robert away

129 Liz: Yes, you've got to go haven't you?

130 Robert: I'm not well.

131 Liz: .. Oh this is terrible because Robert doesn't want to go away and leave this problem.

132 Robert: No I don't.

133 Liz: And I don't want him to either. But you're going to have to go, I can tell. You can't arrive late. (to visitor) That's okay. It's not your fault.

134 Visitor: What are you thinking Robert?

135 Robert: I'm really annoyed with this problem. Umm, I'm trying to find the tangent to the curve $x^2 + 1$ which goes through the origin.

136 Visitor: Ah hmm.

137 Robert: And erh I know the gradient is $2x$. .. So whatever the point that the tangent goes through, the gradient is double the x coordinate. .. And, I'm having difficulty remembering how to tie this in with the erh $y - y_0$ is m into $x - x_0$ equation. And I'm having difficulty remembering what I need to put into y_0 and x_0 to find this. Umm no I'm getting a bit annoyed with that. I tried putting t in but that didn't seem to help. Umm and I tried saying that, well it must be a linear line to this tangent and the c bit of it must be zero. And I tried finding the intersection of y as mx and y as $x^2 + 1$ Umm so I've got x in terms of m which hasn't greatly helped, and you've got $m = \frac{x^2 + 1}{x}$, $x + \frac{1}{x}$, (writes this) isn't helping me very much either.

138 Liz: Hmm.

139 Robert: Hmm.

140 Liz: This is a very hard problem. You're going to have to go

- 1 Sam: Can I do whichever questions I like?
- 2 Liz: Yes. Do you want to try that one.
- 3 Sam: I'll start with five. (*Question 5 reads 'Find the equation of the line with gradient M passing through the point (p, q)*)
- 4 Liz: Okay. Would you like to work on there. (*Sam writes $y - y_1 = m(x - x_1)$, $y - q = M(x - p)$, $y = q + Mx - Mp$*) Ah hmm. Can you tell me where that line crosses the y -axis?
- 5 Sam: Umm where will it cross the y -axis. (*he draws a diagram showing a straight line with positive gradient and y -intercept*) I know it's a straight line. Oh .. that's the point, umm, when .. when x is 0 so at the point $(0, q - Mp)$ No that's wrong. Hmm
- 6 Liz: Why did you decide that was wrong?
- 7 Sam: Umm well because we know if it crosses the y -axis then x is at that point x is 0
- 8 Liz: yes
- 9 Sam: so this point q here is y . So if we replaced all the x s by 0, that will leave us - ah, I think it's that (*writes $(0, q)$*)
- 10 Liz: Ah hmm. Try doing me a diagram of the original situation.
- 11 Sam: No I'm sorry. I think it's $q - Mp$, that's right. Sorry. .. I was getting confused because that umm .. this is going to be 0 here
- 12 Liz: ah hmm. Right. Could you put, you started sketching a diagram there, can you put the point (p, q) somewhere on it. (*Sam marks the point (p, q) on his line in the first quadrant*) Okay. Now does it make sense that this point here (*pointing to the y -intercept*) is $(0, q - Mp)$?
- 13 Sam: Umm .. well if we, if we looked at it from - by drawing the right angled triangle
- 14 Liz: yes
- 15 Sam: to get the gradient (*he draws in a right-angled triangle with two sides parallel to the axes and with vertices at (p, q) and the y -intercept*), we know that this is .. q (*he marks the height of the triangle as q*). This is p . (*indicates the base of the triangle*)
- 16 Liz: Umm .. not sure that that's q . From here up to there. (*indicates the distance from the right-angle to (p, q)*)

- 17 Sam: Umm hang on. What am I doing? Sorry, what is it we're trying to get to? Umm this bit here? (*points to 'q - Mp'*)
- 18 Liz: Yes, I'm just asking you to look at a diagram and see if you can confirm whether or not that's right by looking at the diagram.
- 19 Sam: Right, so that is going to be p . Ah so we know this umm, this point is $(0, y)$. (*marks these coordinates at the y-intercept*)
- 20 Liz: Yes, I wouldn't call it y . I think that might be a bit confusing. It's $(0, \text{something})$ isn't it and we're trying to check out whether the something is $q - Mp$.
- 21 Sam: Yes so we could find a way of expressing that length here (*pointing to the hypotenuse of the triangle*) and erh
- 22 Liz: yes that would be messy. I would come back to this length (*pointing to the height*). You were, you were saying that you were going to draw these lines in so that you could think about the gradient.
- 23 Sam: Hmm.
- 24 Liz: The gradient of this line is M , so what is the relationship between that length and this length? (*pointing to the base and height of the triangle*)
- 25 Sam: Is it the tangent of that angle? No because that's (*inaudible*)
- 26 Liz: Yes it is but that's not particularly helpful.
- 27 Sam: Hmm. .. You see I'm not used to drawing it out as a triangle like that.
- 28 Liz: Ah hmm.
- 29 Sam: I usually use the formula.
- 30 Liz: Hmm. Well tell me about the formula for a gradient then.
- 31 Sam: For a gradient it's $y - y_1, y_2 - y_1$ (*writes $\frac{y_2 - y_1}{x_2 - x_1}$*)
- 32 Liz: Right. And what do those, what does that represent? And what does that represent?
- 33 Sam: So that could be $q - y_1, p - 0$.
- 34 Liz: So that's that height and that width?
- 35 Sam: Hmm.

- 36 Liz: Can you remember where, where did this formula for the gradient come from? Umm draw me a line which has got a gradient of 2. (*Sam draws a diagram showing the line $y = 2x$ and marking the point (2, 4)*) Okay, how do you know that's got a gradient of 2?
- 37 Sam: That's umm by knowing the co-efficient of x . Knowing that the co-efficient of x is 2. The co-efficient of x is the gradient.
- 38 Liz: Right. Okay. Umm the, if you look at this stretch of line, and you've gone from the origin to the point (2, 4), you've gone up 4 and you've gone across 2.
- 39 Sam: Right.
- 40 Liz: Yes? So this distance is twice that distance which is another way of thinking of the fact that the gradient is 2. It goes up twice as much as it goes across.
- 41 Sam: So if it's M it will go, this will be M, Mp , no (*pointing to the height of the triangle*)?
- 42 Liz: Yes.
- 43 Sam: Yes.
- 44 Liz: Yes that's right. I mean if it was a gradient of 2, this would be $2p$ wouldn't it?
- 45 Sam: Yes.
- 46 Liz: A gradient of 3 it would be $3p$, a gradient of M it's Mp . Okay. So we've got that height now.
- 47 Sam: .. Right, so, if we wrote this length here (*referring to the hypotenuse*), we could write it as $p^2 + Mp^2$ - would that be all squared? No -
- 48 Liz: Remind me what it is you want to know now.
- 49 Sam: Umm to confirm this point. If we could find a way of finding the length, writing the length of this
- 50 Liz: umm, .. it wouldn't help much.
- 51 Sam: No. What it is is we need to see if this point, the coordinates satisfy the equation.
- 52 Liz: ... Yes. ... Yes you could do that and it would be a way of checking. Umm . but what I'm asking you to do is to look, is to use the diagram. This distance here (*pointing to the distance along the y-axis from the origin to the intercept*) is the one we're trying to find isn't it? That's the y coordinate of that point. That's what

we're interested in. It's the same as this distance (*pointing to the vertical distance from the x-axis to the right-angle in the triangle*).

53 Sam: Hmm.

54 Liz: Which is this one - (*the vertical height of (p, q) above the x-axis*)

55 Sam: .. minus that one (*the height of the triangle*).

56 Liz: Yes

57 Sam: So that's p (*the base of the triangle*)

58 Liz: hmm, ... what's this length (*the vertical height of (p, q) above the x-axis*)?

59 Sam: That's, which one? That one?

60 Liz: Yes.

61 Sam: $mp +$ umm y .

62 Liz: Yes easier - easier way of doing it though. Look at the coordinates of this point ((p, q)).

63 Sam: Oh it's q .

64 Liz: Hmm.

65 Sam: .. (p, q) ? That's why it's $mp - q, q - mp$ Yes, because we're taking away that bit, right. .. Hmm ...

66 Liz: Yes. Okay.

67 Sam: Not brilliant actually.

68 Liz: Well it's interesting that you didn't have any difficulty answering the question. Right. You did what that question asked you straight away, just by remembering a formula, putting it in. Umm but if I ask you something slightly different about the question, you have to think a bit harder.

69 Sam: I noticed that. I'm not so good

70 Liz: Just go back to basic understandings.

71 Sam: Yes when I'm faced with a problem .. and umm problems with letters

72 Liz: ah hmm

73 Sam: or any problem that I haven't umm looked at before that I can't just bang the formula in or something like that

- 74 Liz: Hmm. Well everybody has that problem, and umm a problem of a type that you haven't done before is always going to be harder.
- 75 Sam: Yes.
- 76 Liz: You've had a look through these. There is a number six which isn't written down on one of these umm. ... Can I get you to try that one and then I'll ask you number six. *(I give him question 3 which reads 'Sketch $y = x(x - a)$ ')*
- 77 Sam: I know that goes through the origin. And, yes that goes through the origin because if you put in $x = 0$, y will be 0 as well. *(he draws a parabola with its vertex at the origin)*
- 78 Liz: Ah hmm.
- 79 Sam: Umm a , .. and when $x = a$, y is 0. Hmm. It's also going this way because the co-efficient of x^2 positive.
..... Hmm. Can you have such a thing as that kind of parabola? *(he draws a very flat-bottomed U shape)* With an x^2 you can't can you?
- 80 Liz: No it would have to be a quartic to be sort of squared off like that. What's this telling you? *(pointing to where Sam has written 'when $x = a$, $y = 0$ ')*
- 81 Sam: .. That if x was a , say anywhere along the point a
- 82 Liz: ah hmm
- 83 Sam: that point y would be 0.
- 84 Liz: Right.
- 85 Sam: .. Right. Hmm. .. That's got to be something like that *(he draws a parabola passing through the origin and $(a, 0)$)*
- 86 Liz: Okay. Umm can you tell me any assumptions you've made about a ?
- 87 Sam: .. a can be positive or can it be negative? Hang on a sec, no it's got to be positive.
- 88 Liz: According to your diagram it has.
- 89 Sam: Well according to this as well because if you had $-a$, .. $-a -a$ would give you $-2a$ wouldn't it?
- 90 Liz: Well suppose a was -2 .
- 91 Sam: A yes it would be $- -2$ wouldn't it?
- 92 Liz: Yes.

- 93 Sam: So it can be neg, it can be a negative number. *(he draws in another parabola showing $(a, 0)$ on the negative x -axis)*
- 94 Liz: Right, okay. What's umm what effect does it have in general if you alter the value of a ?
- 95 Sam: .. It's just going to widen out the umm the parabola.
- 96 Liz: Ah hmm.
- 97 Sam: The parabola's always going to go through the origin no matter what a is so it would either be more squashed together or more spread out.
- 98 Liz: Right. Could, would you be able to tell me about what happens to the turning point if we alter the value of a .
- 99 Sam: Umm as a gets larger the turning point, the y coordinate gets smaller somehow, .. or would it stay the same? Could I try it with two points, with a point,
- 100 Liz: h m m
- 101 Sam: a value for a Just need to differentiate *(he writes $a = 2$, $y = x(x - 2)$, $y = x^2 - 2x$, $\frac{dy}{dx} = 2x - 2$, $2x - 2 = 0$, $x = 1$. $a = 4$, $y = x(x - 4)$, $\frac{dy}{dx} = 2x - 4$, $x = 2$)* Umm what is this telling me? that's not telling me anything *(he crosses out all of his working)*
- 102 Liz: What is it telling you?
- 103 Sam: Well it's just giving me the x point, x coordinate.
- 104 Liz: Okay and what, what do you want.
- 105 Sam: Ah, y oh yes, if I sorry, if I put the x coordinate in and then 2 times then it's 0.
- 106 Liz: What are you putting in?
- 107 Sam: .. x equals 4.
- 108 Liz: .. Where's that coming from?
- 109 Sam: No, x equals 2, sorry, that'd be *(he writes $y = 2(2 - 4) = -4$, $y = 1(1 - 4)$, $y = -3$)*
- 110 Liz: Umm this is the case where $a = 2$ isn't it?
- 111 Sam: Ah hmm.

- 112 Liz: You've got $x = 1$ there.
- 113 Sam: Oh yes. $1 - 2$ (he replaces the 4 in $y = 1(1 - 4)$ by a 2 and changes the final answer to -1) that's -1 , yes that's what I thought. As umm as a gets further away from the origin umm the turning point is going to get closer to the y, x -axis.
- 114 Liz: ... Is it?
- 115 Sam: no. Further away sorry.
- 116 Liz: Hmm. It actually gets deeper doesn't it?
- 117 Sam: Yes.
- 118 Liz: Okay that's great. I think we've plumbed the depths of that question. Have a go at this one (I give him question 6 which reads 'Find the equation of the tangent to the curve $y = x^2 + 1$ which passes through the origin').
- 119 Sam: Is this the tangent which passes through the origin?
- 120 Liz: Yes.
- 121 Sam: (writes $\frac{dy}{dx} = 2x, 2x = 0$) 1. Oh. Why did I go through all that? I knew that. .. Because I differentiated
- 122 Liz: What you were finding the turning point?
- 123 Sam: Yes I knew that.
- 124 Liz: Yes.
- 125 Sam: Just shifted up one and two possibilities (he has drawn a diagram showing the curve and now adds two tangents through the origin) there (he writes $\frac{dy}{dx}(x^2 + 1) = 2x, y = 2x, y = -2x$) Umm is that right? No it's not.
- 126 Liz: Where did it come from?
- 127 Sam: Well to get sorry umm it's not the gradient function is it? No (he writes $y - y_1 = m(x - x_1)$)
- 128 Liz: Is this the equation of the tangent you're doing?
- 129 Sam: Yes.
- 130 Liz: And, and what is, what are x_1 and y_1 ?
- 131 Sam: I think m is 2. .. Umm you - to find x umm when you find it for oh I've forgotten how to do this now.

- 132 Liz: .. You haven't forgotten how to do it. You've never done a question like this.
- 133 Sam: .. Haven't I?
- 134 Liz: No.
- 135 Sam: I thought there was one in the book umm which passed, goes, umm says goes through one, a point and
- 136 Liz: oh yes possibly, yes
- 137 Sam: I think there might have been one.
- 138 Liz: I think I probably didn't ask you to do it but maybe you did it anyway.
- 139 Sam: I know it's, I know I've definitely got to use that somewhere
- 140 Liz: Right.
- 141 Sam: Which
- 142 Liz: You've put the -
- 143 Sam: You get the gradient.
- 144 Liz: - the gradient is 2 there (*Sam has written $y - 1 = 2(x - x_1)$*) . Where did that come from?
- 145 Sam: From here. (*indicating $\frac{d}{dx}(x^2 + 1) = 2x$*)
- 146 Liz: Well it's $2x$.
- 147 Sam: $2x$, umm yes. .. It's not at $x = 0$ because that's not a minimum point.
- 148 Liz: Well $x = 0$ is that point isn't it? (*pointing to the vertex of the curve in Sam's diagram*)
- 149 Sam: What it,
- 150 Liz: ... If you're talking about a point on the curve.
- 151 Sam: Yes.
- 152 Liz: And the tangent certainly doesn't touch at that point.
- 153 Sam: No. So, hmm. I'm stuck.
- 154 Liz: .. Tell me about this. (*pointing to the point of intersection between the curve and the tangent*)

- 155 Sam: Umm coordinates, the point of intersection. If you could work them out then you could work out umm the gradient.
- 156 Liz: Ah hmm.
- 157 Sam: And then work out the equation.
- 158 Liz: Right. But your problem is that you don't know what that point is.
- 159 Sam: Yes.
- 160 Liz: Okay. Well pretend you do. *(he labels the point of intersection (x, y))*
- 161 Sam: That's the gradient *(writes $\frac{y-0}{x-0}$)* Oh that's not going to get me anywhere is it?
- 162 Liz: .. You were just going to write down $y - y$ there, were you? *(he is about to start substituting x and y for x_1 and y_1 in $y - y_1 = m(x - x_1)$)*
- 163 Sam: Hmm.
- 164 Liz: Hmm. Well the, the only reason why that is becoming a problem is that you're using y in two different ways there.
- 165 Sam: Hmm.
- 166 Liz: .. In, in this equation
- 167 Sam: That's $\frac{y}{x}$ the gradient.
- 168 Liz: Yes. There are a lot of x s and y s in this problem. ... You might make life easier for yourself if you chose something different for that point.
- 169 Sam: a and b maybe *(he replaces (x, y) by (a, b))*
- 170 Liz: Hmm $\frac{b}{a}$ isn't it for the gradient. *(Sam has written $\frac{a}{b}$)*
- 171 Sam: Umm so that's the equation *(he has written $2x = \frac{b}{a}$)*
 $y - b = \frac{b}{a}(x - a), y - b = b(x - 1), y = b + bx - b$
- 172 Liz: ah hmm
- 173 Sam: if we took it at the point $(0, 0)$ it would be that - would just give us $0, y = 0$, yes. .. The point 1 - is that - ? .. I took it the point 1,

no, have we actually got to write this with umm numbers?
 $y = 2x + \text{something}$

174 Liz: Yes.

175 Sam: .. Yes. I'm sure I've done this type of question before.

176 Liz: Go back to what you were saying about (0, 0).

177 Sam: Umm .. what you mean using the point

178 Liz: h m m

179 Sam: in the equation?

180 Liz: Yes. No I see what you mean now when you say you just get $y = 0$. Yes okay. Umm I'm going to give you a couple of hints. There are two other things that you know about this point that, well this point and this line. One of them is that this point (a, b) actually lies on this curve as well as lying on the line.

181 Sam: Yes.

182 Liz: And the other one is that the gradient of this line is the same as the gradient of the curve at that point.

183 Sam: It's not going to help me much.

184 Liz: Umm this equation here, $2x = \frac{b}{a}$, where did you get that from?

185 Sam: If we differentiate this

186 Liz: yes

187 Sam: oh I know. (he writes $2a = \frac{b}{a}$)..... It's $b = ..$ (he writes $b = a^2 + 1$) times it by a on both sides and you get (he writes $2a^2 = b$)

188 Liz: ah h m m

189 Sam: so that gives us $2a^2 = a^2 + 1$, (writes $2a^2 = a^2 + 1$, $a^2 = 1$, $a = 1, -1$) h m m then we can use that (he writes 'when $a = 1$, $b = 2$, gradient = 2, when $a = -1$, $b = 2$, gradient = -2, $y - y_1 = m(x - x_1)$, $y = 2x$, $y = -2x$) That's what I thought, it comes back to this bit doesn't it? (referring back to where he has written $y = 2x$, $y = -2x$ at the beginning of his working)

190 Liz: Hmm.

191 Sam: ... So I got it from, straight from here, yeah

- 192 Liz: Hmm. Umm, you got it straight from there and then you decided it was wrong.
- 193 Sam: Yes. I don't know why it, I knew it was right because it goes through the origin.
- 194 Liz: Ah hmm.
- 195 Sam: So you can't have a +1 on or anything like that.
- 196 Liz: Yes, but why did it have to be 2? Why $2x$?
- 197 Sam: Umm why $2x$?
- 198 Liz: You see supposing I'd given you the same question but I'd said the curve is $y = x^2 + 2$ Do you think the answer would be the same?
- 199 Sam: No. It would be ..
- 200 Liz: It would be steeper.
- 201 Sam: It would be steeper.
- 202 Liz: But if it had been $x^2 + 2$ your working here (*pointing to his first bit of working* $\frac{d}{dx}(x^2 + 1) = 2x$, $y = 2x$, $y = -2x$) would have been the same.
- 203 Sam: Hmm.
- 204 Liz: So if at that stage you'd have said 'oh it's $y = 2x$ ' you would have been wrong.
- 205 Sam: Yes.
- 206 Liz: .. So when you made this step, I think it was a bit of inspired guess work.
- 207 Sam: Hmm yes.
- 208 Liz: .. So you do actually have to go through all the working that you did there to get the answer. Umm could you, this is a difficult question, I haven't found anybody of your age who can do it without a bit of help. Umm can you say why you think it's difficult?
- 209 Sam: It's difficult because this point you, you just don't know, haven't got any idea. I mean, .. yes, what, that's, what is it that makes it difficult? .. The problem was finding the gradient. That was the greatest problem.
- 210 Liz: Ah hmm.

- 211 Sam: I guess
- 212 Liz: There was a moment when you'd written $2x = \frac{b}{a}$ here and I said where did this $2x$ come from. And you said, oh yes, and you changed it into $2a$.
- 213 Sam: Hmm.
- 214 Liz: Why did you do that? What made you think of that?
- 215 Sam: Because umm the gradient of, by differentiating you know that the gradient of the curve is $2x$.
- 216 Liz: Ah hmm.
- 217 Sam: So if, if this point is a , the coordinate, x coordinate is a , you know the gradient's going to be $2a$. So you can replace x here by a .
- 218 Liz: Right. And can you remember what it was that brought that to mind? Why did you suddenly think of it?
- 219 Sam: .. Umm well just by looking at that.
- 220 Liz: Right. .. Okay.
- 221 Sam: So yes I guess that's the thing really. .. But if I'd seen that may be from the start, that umm, just looking for this point and finding erh some erh relationship between this gradient here and that
- 222 Liz: ah hmm
- 223 Sam: if I'd focused on that, it might have avoided going round in circles.
- 224 Liz: Right. .. Okay. That's great. Thank you very much.

- 1 Liz: There's six questions. Umm you're certainly not going to have time to do all of them I wouldn't think. Have a look through them and see if there's any in particular you'd like to start with Some of them will be quite familiar I think.
- 2 Tommy: Yes .. Erh number 6. (*Number 6 reads "Find the equation of the tangent to the curve $y = x^2 + 1$ which passes through the origin"*)
- 3 Liz: Okay. Would you like to write on there if you want to write anything.
- 4 Tommy: Shall I write out the question?
- 5 Liz: Sorry?
- 6 Tommy: Shall I write out the question?
- 7 Liz: No, no, no. Don't write out the question. Just any working you want to do.
- 8 Tommy: .. Umm, .. first which passes through the origin?
- 9 Liz: ah hmm
- 10 Tommy: Is that the point (0, 0)?
- 11 Liz: yes.
- 12 Tommy: So, so if I differentiate this
- 13 Liz: ah hmm
- 14 Tommy: so, if I take that, that would equal $2x$, (*writes*
$$\frac{d}{dx}(x^2 + 1) = 2x$$
).... it goes through the origin, so x should be zero.
- 15 Liz: Hmm.
- 16 Tommy: Obviously If I do a sketch of this equation it should be this .. (*he draws a parabola with its vertex at (0, 1) and a tangent to it passing through the origin*)
- 17 Liz: ah hmm
- 18 Tommy: It goes through the origin and that's the equation I'm trying to find
- 19 Liz: right

- 20 Tommy: so, so we know that the equation I'm trying to find should be $y = mx + c$ (*writes $y = mx + c$*)
- 21 Liz: ah hmm
- 22 Tommy: m is the gradient, and c is the y intercept
- 23 Liz: right
- 24 Tommy: so it will be $y = mx + 0$ (*writes $y = mx + 0$*) um, so $2x$ differentiated will come out to replace the mx (*writes $2x = m$*).
- 25 Liz: Say that again.
- 26 Tommy: You replace the mx with this $2x$. (*writes $2x = mx$*)
- 27 Liz: Right. Replace it?
- 28 Tommy: Yes.
- 29 Liz: Why are you doing that?
- 30 Tommy: Umm because that's the gradient function
- 31 Liz: ah hmm
- 32 Tommy: (*inaudible*)
- 33 Liz: What do you mean by gradient function?
- 34 Tommy: Gradient function, .. the $2x$, if you have any point on the curve
- 35 Liz: ah hmm
- 36 Tommy: so they're like $x = 2$, then substitute into this equation $2x$, 2 times 2 is 4, that means the gradient is 4 when x is equal to 2
- 37 Liz: ah hmm.
- 38 Tommy: on this curve
- 39 Liz: Right. Okay. So how does that tie in with this mx ?
- 40 Tommy: Well the gradient is going to be equal to the curve's gradient
- 41 Liz: ah hmm
- 42 Tommy: at the point of intersection.
- 43 Liz: Right.
- 44 Tommy: So that will be $2x$ is equal to mx . (*writes $2x = mx$ again*)

- 45 Liz: Ah ha. Umm .. mx is the gradient then is it?
- 46 Tommy: No. (*inaudible*) Is that right? (*he crosses out the last two equations*)
- 47 Liz: No.
- 48 Tommy: Well,
- 49 Liz: But do you know what's wrong with it? What's the gradient of this line?
- 50 Tommy: This line? (*pointing to the tangent*)
- 51 Liz: Hmm.
- 52 Tommy: m .
- 53 Liz: Hmm. Not mx ?
- 54 Tommy: No. $2x$ is equal to m .
- 55 Liz: Ah hmm. Where is that equation true?
- 56 Tommy: At the point of intersection.
- 57 Liz: Right. Okay.
- 58 Tommy: So I can rearrange that to find so $y = 2x + 0$. (*writes $y = 2x + 0$*)
- 59 Liz: What's that?
- 60 Tommy: This?
- 61 Liz: Yes, what's that, the equation? Where did you get that from you've written down?
- 62 Tommy: In the m is equal to $2x$.
- 63 Liz: Ah hmm.
- 64 Tommy: Then substitute that into the general equation
- 65 Liz: yes
- 66 Tommy: $y = mx$ becomes $y = 2x$.
- 67 Liz: y is equal to mx (*stressing the x*).
- 68 Tommy: Oh yes. that means um substitute that in to $y = .. 2x$ times x (*writes $y = (2x)x$*)
- 69 Liz: h m m

- 70 Tommy: That can't be right.
- 71 Liz: Why not?
- 72 Tommy: Because if you multiply this out, $2x$ times x becomes $3x$
- 73 Liz: $2x^2$
- 74 Tommy: that's it $2x^2$. And it can't be a squared because you actually asked if it would be linear.
- 75 Liz: Right. Okay. I agree with you then. That's not right.
- 76 Tommy: .. So, you know that m is equal to $2x$ at the point of intersection
- 77 Liz: right
- 78 Tommy: .. so we know that where x is equal to zero y is equal to zero
- 79 Liz: hmm Would you like some help?
- 80 Tommy: Yes please.
- 81 Liz: You only need to ask. Umm this equation here is the important one. Umm it's rather a difficult one to understand, $2x = m$, and the key to its difficulty is that it's only true at the point where the tangent touches, right, which you've already told me twice. Umm but you're not quite sure what to do with that information. Umm the problem that you're having so far is that you've got this equation, $2x = m$, now m is the gradient of the tangent and it's fixed, if only we could find out what it was, it represents an actual number. But x plays lots of different roles in this question. It appears in the equation of the curve, $y = x^2 + 1$. And it doesn't take any particular value in that equation. It could take any value. It occurs in this equation (*pointing to* $y = mx + c$), the equation of the tangent that we're trying to find, and there it could take any value. Umm so if you try and use it in here to stand for a particular value, then it becomes confusing. Umm so if I were you, I think the key to making progress on here is to decide to call the value of x at this point something else, and work with that. So give it another name.
- 82 Tommy: c .
- 83 Liz: Hmm, c 's a bit difficult.
- 84 Tommy: Yeah - d .
- 85 Liz: d , okay fine.
- 86 Tommy: $2d = m$ (*writes* $2d = m$)

- 87 Liz: Right. So d is the x co-ordinate at this point? Right. You were trying to find a value for m in the end aren't you? Umm and we haven't got much information about it yet. So we need to think of some more things we can say about d and m .
- 88 Tommy: d would be m divided by 2
- 89 Liz: Ah hmm.
- 90 Tommy: By rearranging this equation
- 91 Liz: Yes. Try bringing in the y co-ordinate at that point as well.
- 92 Tommy: The y co-ordinate How will I do that?
- 93 Liz: Well, first you need to give it a name.
- 94 Tommy: The y co-ordinate?
- 95 Liz: Hmm.
- 96 Tommy: Umm e .
- 97 Liz: Okay. Now what do you know about it?
- 98 Tommy: Hmm, at the point of intersection?
- 99 Liz: Hmm.
- 100 Tommy: It's greater than 1
- 101 Liz: Yes
- 102 Tommy: I'm stuck
- 103 Liz: Well how is e connected with d ? You know that d and e are the co-ordinates of this point, and that's a point which lies on the tangent and it's a point which also lies on the curve. So you should be able to write me down some connections between d and e .
- 104 Tommy: I've forgotten what d is now! d is the x co-ordinate isn't it?
- 105 Liz: Yes.
- 106 Tommy: So the point of intersection would be (d, e) (*adds this to his diagram*)
- 107 Liz: ah hmm

- 108 Tommy: so that would be $e^2 + d^2 = .$ Using Pythagoras to work out .. no, you can find the equation of the gradient, the equation of the gradient of this line, would be e over d
- 109 Liz: ah hmm, .. and what is the gradient of that line?
- 110 Tommy: 2. No, $2d$. (writes $\frac{e}{d} = 2d$)
- 111 Liz: $2d$? Okay.
- 112 Tommy: .. So e is equal to $3d$ (writes $e = 3d$)
- 113 Liz: $2d^2$
- 114 Tommy: $2d^2$?
- 115 Liz: Ah hmm. $2d$ times d .
- 116 Tommy: Oh yes ... So e is equal to $2d^2$ (writes $e = 2d^2$)
- 117 Liz: So you've got one equation linking e and d , and you got that by knowing that this line which goes through the origin and has a particular gradient, goes up to this point. Okay. Focus also on this curve, because this point here lies on this curve as well. What will that tell you about d and e ?
- 118 Tommy: Say that again.
- 119 Liz: The fact that that point lies on the curve, what would that tell you about d and e ?
- 120 Tommy: The .. d and e satisfy, the co-ordinates d and e will also satisfy this equation.
- 121 Liz: Right.
- 122 Tommy: d^2
- 123 Liz: ah hmm
- 124 Tommy: plus 1 (writes $d^2 + 1 = e$)
- 125 Liz: ah hmm
- 126 Tommy: (writes $e = d^2 + 1$, $e = 2d^2$).....Take away this one makes it negative d^2 plus 1 (writes $-d^2 + 1 = 0$), so therefore d is equal to plus or minus 1.
- 127 Liz: Ah hmm.
- 128 Tommy: .. Plus 1 it's got to be

- 129 Liz: Umm why do you get a minus 1 as an answer as well?
- 130 Tommy: Well because you root d^2 and if you square a negative 1, it becomes 1
- 131 Liz: hmm. So why are you rejecting the negative one answer?
- 132 Tommy: Well on the graph it's obvious that d 's going to be positive.
- 133 Liz: You don't think your point could be over here? (*pointing to the negative side of the curve*)
- 134 Tommy:I didn't consider that.
- 135 Liz: Think what?
- 136 Tommy: I didn't consider that.
- 137 Liz: No. Well the question is badly worded because it says the equation of "the tangent", but in fact there are two, because this curve is symmetrical. You could have a line coming through here as well. So both answers are okay.
- 138 Tommy: Fine
- 139 Liz: Right. Okay. Can you finish the question off? .. What does it ask you for?
- 140 Tommy: The equation. $y = ..x$, $m = 2d$.. d times, 1 times 2, x , so y is equal to $2x$.
- 141 Liz: Ah hmm. And what about this one?
- 142 Tommy: That will be y is equal to negative $2x$.
- 143 Liz: Okay. Have you got your graphic calculator with you? (*he reaches into his bag for it*).. Never go anywhere without it?
- 144 Tommy: I had Geography today.
- 145 Liz: Oh right. Have a look at those graphs.
- Tommy plots $y = x^2 + 1$, $y = 2x$ and $y = -2x$ on his calculator and appears satisfied by the result.*
- 146 Liz: Right. Now, that was a very hard question. I think it's the hardest out of the six. I don't know why you decided to do that one first (laughs).
- 147 Tommy: I thought it looked easy.
- 148 Liz: You thought it looked easy. Umm can you tell me why it turned out to be hard even though it looked easy.

- 149 Tommy: Umm I got confused with all the different variables
- 150 Liz: Hmm. Can you say a bit more about that?
- 151 Tommy: Replacing the x co-ordinate with d and the y co-ordinate with e , I got confused
- 152 Liz: Hmm.
- 153 Tommy: *inaudible*
- 154 Liz: Do you think it would have been easier if you'd left them as x and y ?
- 155 Tommy: Umm, I don't know really.
- 156 Liz: Well you did try that to begin with didn't you?
- 157 Tommy: (*inaudible*) x on both sides of the equation
- 158 Liz: Ah hmm. At one stage you had written that (*pointing to $y = 2x$*) down, which is actually the right answer isn't it?
- 159 Tommy: Because I had $2x$ is equal to mx .
- 160 Liz: Yes.
- 161 Tommy: The x cancelled out leaving 2 equal to m
- 162 Liz: Hmm. Umm what do you think of that as a method of solution?
- 163 Tommy: This one?
- 164 Liz: Hmm.
- 165 Tommy: It's more obvious.
- 166 Liz: Right. .. Try umm the same question but for the curve, $x^2 + 2$.
- 167 Tommy: $x^2 + 2$ (*inaudible*) (*he draws a graph of $y = x^2 + 2$ showing two tangents through the origin*)..... differentiate and it's $2x$ again
- 168 Liz: ah hmm
- 169 Tommy: so .. the only point of intersection between the equation $y = mx + c$ mx should equal 2.
- 170 Liz: What were you saying?
- 171 Tommy: Umm I was saying something similar to that
- 172 Liz: right

- 173 Tommy: be, would equal $2x$, at the point of intersection
- 174 Liz: Right. Does that lead you to $m = 2$ again?
- 175 Tommy: Yes.
- 176 Liz: And is that wrong?
- 177 Tommy: It should be because the one you set on this card was 2.
- 178 Liz: Ah hmm. .. So $y = 2x$ can't be the tangent.
- 179 Tommy: No.
- 180 Liz: Right. Do you want to draw $x^2 + 2$ and just confirm that for yourself. (*Tommy plots $y = x^2 + 2$ on his calculator*)..... Yes. Okay. So doing this, it works exactly the same on this question as it did on this question, but it's not right. Umm so that suggests to me that there's something wrong with the method. Can you see what the problem is?
- 181 Tommy: No.
- 182 Liz: Do you remember when I asked you if you wanted some help, and we were talking about this equation, and I was saying something about the x , can you remember what that was? No? .. Umm I was saying that the x in this equation here (*pointing to $2x = m$*) means the x at this point, and not any other x . Umm but in mx the x means any x .
- 183 Tommy: Oh yes.
- 184 Liz: So if you write an equation like $2x = mx$, you can't cancel the x s out, because they're not the same x .
- 185 Tommy: Oh yes.
- 186 Liz: Umm which is why it's very confusing to call this x and why I suggested you call it d instead.
- 187 Tommy: Yes.
- 188 Liz: ... So if you wanted to do this question, $x^2 + 2$, you would really have to use the same sort of method that you did here. Umm because this method is a false method. By coincidence it gives you the right answer in the first case, but it won't if you change the question. Okay I don't think I'll ask you to finish that off because it's just the same as the one you've just done. Umm let's have a look at another question. Can you try number three. (*Number 3 reads "Sketch $y = x(x - a)$ "*)
- 189 Tommy:I think I'll try factorising it

- 190 Liz: what does that mean?
- 191 Tommy: Well expand out the brackets, x , x take a in brackets becomes $x^2 - xa$
- 192 Liz: ah hmm
- 193 Tommy: x is if you consider when x is equal zero
- 194 Liz: ah hmm
- 195 Tommy: because ... then it will cross the y -axis at zero
- 196 Liz: ah hmm
- 197 Tommy: because zero take any number times, multiplied by zero will give zero. I know it will cross the origin
- 198 Liz: ah hmm
- 199 Tommy: and if x is equal to a , a take a becomes zero, that will be the point a
- 200 Liz: ah hmm
- 201 Tommy: it will be positive because x , the x take a a has to be positive to make the sum of that zero. .. So .. well you can see when you times it out it will have a power of 2, that's a quadratic. It's a positive quadratic.
- 202 Liz: Ah hmm.
- 203 Tommy: So, I'm not drawing this very well (*he sketches a parabola with positive orientation passing through a point $(a, 0)$ on the positive x -axis and through the origin*)
- 204 Liz: Okay, fine. Umm what happens to this curve if you vary what the value of a is?
- 205 Tommy: If you vary it, it can either expand out
- 206 Liz: what do you mean by that?
- 207 Tommy: well this cross over (*indicating the point $(a, 0)$*) will move along slightly
- 208 Liz: right, ah hmm. You said it could either do that or.
- 209 Tommy: Or, we take the a goes towards and becomes negative, it could either go to the negative
- 210 Liz: ah hmm
- 211 Tommy: like this

- 212 Liz: What if a does become negative?
- 213 Tommy: Hmm the root should become negative.
- 214 Liz: Ah hmm. Could you draw me what it would look like in that case.
- 215 Tommy: This would change. *(he draws a second parabola crossing the x-axis for a second time at a point on the negative side. This point is labelled $-a$)*
- 216 Liz: .. Right, okay. Umm supposing that a becomes bigger, so you say the root is travelling along here. What happens to the turning point on the curve? Does it go lower or higher?
- 217 Tommy: Higher.
- 218 Liz: Ah hmm. Is that a guess?
- 219 Tommy: Well if you consider the turning point, negative b over $2a$ as a becomes greater, the factor here becomes greater so reducing the fraction
- 220 Liz: Where does this negative b over $2a$ come from?
- 221 Tommy: Oh it's the formula for finding the greatest point on a quadratic.
- 222 Liz: Ah hmm.
- 223 Tommy: Greatest point on a quadratic
- 224 Liz: What's the, what's the quadratic on which that is the turning point?
- 225 Tommy: Pardon?
- 226 Liz: Which quadratic are we talking about?
- 227 Tommy: Umm this one $x^2 - ax$.
- 228 Liz: So what's b then?
- 229 Tommy: .. b should be a . *(laughter)* No b should be, in this equation a is equal to b in this equation
- 230 Liz: right. So b is the co-efficient of x ?
- 231 Tommy: .. Yes.
- 232 Liz: Ah hmm.
- 233 Tommy: If I can change this to, if I change this to be .. b is equal to a , x^2 , that will become negative a over $2d$, which will become a over 2

- 244 Liz: right
- 245 Tommy: $2a$, no, equals So when a becomes, when a goes to a big number, infinity, this fraction, the denominator will become larger, go to infinity
- 246 Liz: What's d ?
- 247 Tommy: d is the coefficient of x^2 .
- 248 Liz: And what's the coefficient of x^2 ?
- 249 Tommy: One.
- 250 Liz: Uhmhm. a over 2 is right, all on it's own, with no d 's involved, because if you look at this, this, the two roots, the turning point is half-way between them, isn't it. Half-way between zero and a which is a half a , a over 2. Umm but that's the x co-ordinate. What I was actually asking you about is does the turning point get lower or higher, as you pull a out?
- 251 Tommy: Higher.
- 252 Liz: Okay, why? Would you like to try a few examples out on the graphic calculator and see whether your conjecture is right? (*Tommy works on the calculator*) So you've got x into x minus 3 and x into x minus 5. Which is which?
- 253 Tommy: Well this one is x times x take 3.
- 254 Liz: Ah hmm.
- 255 Tommy: And this one is x times x take 5.
- 256 Liz: Right.
- 257 Tommy: It seems to have a minimum point roughly lower
- 258 Liz: h m m
- 259 Tommy: lower, as it gets farther away
- 260 Liz: yes certainly in the case of those two it is. Yes What I, what I was asking you about was the, how low down the turning point is, which is the y co-ordinate of the turning point? Do you think you could find out what the y co-ordinate of the turning point is for this curve, $x^2 - ax$?
- 261 Tommy: a over 2 squared, because x is equal to a over 2
- 262 Liz: right

263 Tommy: I think, (*mumbling*) (*writes* $\left(\frac{a}{2}\right)^2 - a\left(\frac{a}{2}\right), \frac{a^2}{4} - \frac{a^2}{2}$)

264 Liz: ah hmm

265 Tommy: (*mumbling* - *writes* $= \frac{2a^2 - 4a^2}{8}, \frac{-2a^2}{8}$)

266 Liz: Ah hmm. So you just cancel that down.

267 Tommy: A quarter, negative a quarter a^2 . (*writes* $-\frac{1}{4}a^2$.)

268 Liz: Right. So what does that tell you about whether the lowest point and whether the minimum point is getting higher or lower?

269 Tommy: Well as a goes to infinity the numerator would have to get bigger.

270 Liz: Ah hmm.

271 Tommy: Moving out would have to go lower.

272 Liz: What will have to go lower?

273 Tommy: Umm the y -coordinate

274 Liz: Right, okay. So as a gets bigger the minimum point gets lower down. Right. And in fact, it gets lower down quite fast because it's a^2 umm so in going from three to five we go from nine to twenty five. It's a quarter of each of those, but it's still going to happen quite rapidly. The umm, you can see from here that the difference between three and five, there's not much compared with the difference between the two minima. Try another one. See if there's any difference (*Tommy works on the calculator*) Hmm, right off the screen

275 Tommy: That's x times x take 7

276 Liz: Right. Okay. Umm have you got a lesson at twelve o'clock.

277 Tommy: No

278 Liz: We'll just do one more then. Can you do number five (*Number 5 reads "Find the equation of the line with gradient M passing through the point (p, q) "*). Time for a clean sheet of paper I think.

279 Tommy: "passing through the point" - I'll sketch that first - the point (p, q) could be anywhere

280 Liz: Ah hmm.

- 281 Tommy: .. Gradient M , any gradient so just a rough sketch (*he draws a diagram showing the point (p, q) in the positive quadrant and a line with negative gradient passing through it*)
- 282 Liz: What choice have you made about M in drawing that diagram?
- 283 Tommy: I've made it a negative, when it could be a positive.
- 284 Liz: Sorry?
- 285 Tommy: It could be positive.
- 286 Liz: Yes .. (*Tommy adds a line with positive gradient through (p, q) to his diagram*) Okay.
- 287 Tommy: So I'm using the formula $y = mx + c$ y would be q , would equal $mx +$, no, $mp + c$ (*writes $q = mp + c$*)
- 288 Liz: ah hmm
- 289 Tommy: rearrange that to find the c which would be q take mp is equal to c (*writes $q - mp = c$*)
- 290 Liz: Ah hmm.
- 291 Tommy: Therefore y is equal to mp plus q take mp (*writes $y = mp + (q - mp)$*) That's, that's the positive gradient.
- 292 Liz: Is that the equation of the line you've got there?
- 293 Tommy: It's the positive equation.
- 294 Liz: All right. .. Why do you say that that's just for the positive one?
- 295 Tommy: Well because I considered it m being positive. I could have made it negative which would have made that q plus mp
- 296 Liz: Hmm. No you don't make m negative by putting a minus sign in front of it. Umm m could be a negative number. Let's say, I mean m might have been negative 2. Umm and if you'd put a minus sign in front of it, that makes it positive.
- 297 Tommy: Oh (laughs).
- 298 Liz: So you don't actually need to do anything different. Umm just by putting m in here, it could be positive or it could be negative, and it doesn't, you don't have to do anything different. So you only need to do it once, even though you've drawn two different diagrams. Umm but tell me what the equation of a straight line is like.
- 299 Tommy: The equation of a straight line?

- 300 Liz: Hmm.
- 301 Tommy: Well it has a constant gradient
- 302 Liz: right
- 303 Tommy: it crosses the y -axis at c , the equation is $y = mx + c$.
- 304 Liz: Right.
- 305 Tommy: And it usually only has one root - it has one root
- 306 Liz: It only has one root - Right okay. So umm look at the equation that you've just given me as your answer. Does that satisfy your conditions? Okay, go back to this form (*meaning* $y = mx + c$) because this is really what you were talking about, wasn't it. You were telling me about what the m was, and what the c was, and so on. Umm what else have we got in this equation apart from the m and the c ?
- 307 Tommy: The x .
- 308 Liz: Ah hmm. And the y .
- 309 Tommy: And the y .
- 310 Liz: Right. Have a look at this one again (*pointing to* $y = mp + (q - mp)$).
- 311 Tommy: There's no x in there
- 312 Liz: Right
- 313 Tommy: That should be x (*he changes the first* p *in* $y = mp + (q - mp)$ *to* x)
.....
- 314 Liz: Ah hmm. Why do you need to change that to x ?
- 315 Tommy: Well we definitely consider p it will only be p , at this point here (*pointing to* (p, q))
- 316 Liz: Right.
- 317 Tommy: But the general equation of a line, we've got to consider x .
- 318 Liz: Okay. Why do you think you got it wrong in the first place?
- 319 Tommy: Well I wasn't paying attention to it
- 320 Liz: Right. Could umm $y = mp + q - mp$ be the equation of a straight line?
- 321 Tommy: If, if the axis was p and q

322 Liz: ah hmm. Yes. But if the axes were x and y , (*Tommy shakes his head*) well it could actually because umm something like $y = 2$ is the equation of a straight line isn't it? You don't need to have x in there for it to be the equation of a straight line. Umm $y = 2$ is parallel to the x axis so x doesn't appear in the equation, so what you had written down to start with was the equation of a straight line, but not the one you wanted. Umm okay. Umm, ... a minute ago I was asking you about minus b over $2a$. And I said to you which curve is this the x co-ordinate of the turning point of. And you told me it was this one, which is quite true, umm but it wasn't what I meant. Umm I meant where did this, where did this come from? Which curve were we talking about when we came up with that formula?

323 Tommy: $ax^2 + bx + c$

324 Liz: Yes. But the umm, .. yes, okay. Right. I think we've finished. Thank you very much Tommy.

Trevor and Paul I 11/2/94

- 1 Liz: Okay, well I've picked out what seemed to be the most important things about this chapter.
- 2 Trevor: Oh, I hate the lot of it. I just can't remember what to do every time I come up to a question I have to go back through about 700.
- 3 Liz: The thing is if you get a better mental image of what is going on in this chapter then you won't need to remember the methods, they will be obvious.
- 4 Trevor: Quite a lot of it I can (*inaudible*), quite a lot of it is to do like with graphs and things like that but I can't see what the graph's going to look like.
- 5 Liz: Yes, well often you don't need to.
- 6 Trevor: Yeah but when you are finding gradients and things like that.
- 7 Liz: Yeah, okay. I'm going to ask you three questions to start with, then we will have a look at some of the ones from their exercises. So, this chapter is all about this equation (*writes* $y = mx + c$) So tell me as much as you can about what this means to you.
- 8 Trevor: What that means to me. That's the equation of a line.
- 9 Paul: Yeah, no.
- 10 Liz: Yeah?
- 11 Paul: It's to find the gradient? No, its not to find the gradient, 'cos the gradient's (*inaudible*)
- 12 Trevor: It can tell you where, where the line crosses the y -axis.
- 13 Liz: Right, which bit tells you that?
- 14 Trevor: c . Well it would tell you the gradient if you knew (*inaudible*)
- 15 Paul: The gradient's m .
- 16 Liz: The gradient's m , okay. So, give me an example of the equation of a line in that form and what it tells you.
- 17 Paul: $y = 4x + 3$.
- 18 Liz: Right, which tells you?
- 19 Paul: It's got a gradient of 4 and it cuts the
- 20 Trevor: y -axis.

- 21 Paul: y -axis at 3.
- 22 Liz: Fine, that's really all you need to know for this chapter.
- 23 Paul: Is it?
- 24 Liz: Yeah
- 25 Trevor: Well that's wonderful
- 26 Liz: Two more questions. If I tell you that that line goes through this point (1, 2) what does that enable you to say about m and c ?
- 27 Trevor: Umm. $y = 2$. If you substitute those numbers in then you can probably work out what the other two are.
- 28 Liz: Okay, could you write something down for me?
- 29 Trevor: $2 = m + c$ $m + c$ there's two unknowns
- 30 Paul: Well you know $m + c = 2$
- 31 Liz: Umhm. Does that actually specify the line? Does it tell you what the equation of the line is?
- 32 both: No.
- 33 Liz: Because, why not?
- 34 Trevor: I don't know. I always thought for the gradient of a line you had to have two coordinates. Then subtract y_2 from y_1 .
- 35 Liz: Okay, if you get just a picture in common sense terms of what I've said about this. This $y = mx + c$ is any straight line.
- 36 Paul: Yeah.
- 37 Liz: If you have got any straight line you can write it in that form.
- 38 Trevor: Yeah
- 39 Paul: Uhummm
- 40 Liz: And all I have told you is it goes through this point.
- 41 Trevor: Yeah
- 42 Paul: Uhummm
- 43 Liz: So that could be a lot of different lines couldn't it. So you wouldn't expect to be able to work out what the equation of the line is just from that one point. If all I say is it's got to go through there
- 44 Trevor: Could be parallel.

- 45 Liz: It could be like that, it could be like that, could be like that, could be any number of things. What else could I tell you so that you would know exactly which line it was?
- 46 Trevor: Another coordinate?
- 47 Liz: Another point, yes.
- 48 Paul: Or the gradient.
- 49 Liz: Or the gradient.
- 50 Paul: Or where it cuts the y
- 51 Trevor: Or x
- 52 Paul: or x axis.
- 53 Liz: In effect, of course, saying where it cuts one of the axes is giving another point.
- 54 Paul: Yeah.
- 55 Liz: But yes, that would be another way of doing it. Okay, so forget that condition, that one's not true anymore. I'm going to say now it goes through $(0, 4)$.
- 56 Trevor: As well as $(1, 2)$.
- 57 Liz: No.
- long pause*
- 58 Paul: It's $(4, 0)$.
- 59 Liz: Yes.
- 60 Trevor: c has to equal 4.
- 61 Paul: c equals 4, because 4 equals $0 + c$
- 62 Trevor: So the gradient is 0.
- 63 Liz: It tells us that c is 4, which is, you could have done that by a slightly different sort of reasoning because, c , you said to me was the point on the y -axis where it cuts.
- 64 Trevor: Yeah.
- 65 Liz: And this point $(0, 4)$ is on the y -axis. It goes through $(0, 4)$ then c is 4. What does it tell us about m ?
- 66 Paul: That it's 0 because

- 67 Trevor: I don't know if it would be 0, cause you are just saying that x is 0. It still could be at an angle
- 68 Paul: We know, we know that $y = 4$, in this particular case and we know that 4 is c , so we know that mx has got to equal 0.
- 69 Liz: Yeah.
- 70 Paul: And the only way mx could equal 0 is if m is 0.
- 71 Trevor: No but
- 72 Paul: No, No, No - because x is 0.
- 73 Trevor: x is zero, so m could be anything
- 74 Paul: Yeah that's it so m could be anything
- 75 Liz: m could be anything, that's right yes. Again I have only given you one point that it goes through. We know it goes through this point here, but I haven't told you what the gradient is, and it could be any gradient. So m could be anything. And again we could decide exactly which line it is by me giving you another point that went through or by me giving you the gradient. Well, you know really if you understood all that there's not much more to it.

Later in the conversation:

- 76 Liz... Um, question 1 What's it asking you to do?
Question 1 reads as follows:
Find the equation of the line with the given gradient passing through the given point.
- | | | |
|----------------|---|--|
| (a) 3, (4, 9) | (b) -5, (2, -4) | (c) $\frac{1}{4}$, (4, 0) |
| (d) 0, (-1, 5) | (e) $-\frac{2}{5}$, ($\frac{1}{2}$, 4) | (f) $-\frac{3}{8}$, ($\frac{22}{5}$, $-\frac{5}{2}$) |
- 77 Trevor: It's asking . 'Find the equation of the line with the given gradient ..' Find the equation of a line
- 78 Liz: Uhuh
- 79 Trevor: given those two
- 80 Liz: So you've got the gradient and you've got one point

- 81 Paul: Uhum
- 82 Liz: Are you confident about a method to do that?
- 83 Trevor: Yes. $y = mx + c$
- 84 Liz: Uhuh
- 85 Trevor: And you just substitute 4 and 9 in for y and x and 3 for m and then you just find out what c is
- 86 Paul: Find what c is
- 87 Liz: Find out what c is - fine, okay.

Trevor and Paul II 11/2/94

- 1 Liz: I picked out what I thought were the important ones - question 4
Question 4 reads 'Write down the equation of the perpendicular bisector of the line joining the points $(2, -3)$ and $(-\frac{1}{2}, 3\frac{1}{2})$ '
- 2 Trevor: Yeah you just find out the gradient and then
- 3 Paul: Find out the gradient and then find out the equation of the line
- 4 Trevor: Then you add the two y s together and divide by 2
- 5 Paul: Yeah you find the gradient and the mid-point and then reverse the gradient - minus 1 over m
- 6 Liz: Uhmhm
- 7 Paul: And put it in with the coordinates of the mid-point
- 8 Liz: Yep. Fine, good. Number five?
Question 5 reads 'Find the equation of the line through $A(5, 2)$ which is perpendicular to the line $y = 3x - 5$. Hence find the coordinates of the foot of the perpendicular from A to the line.'
.....
- 9 Paul: Do you - you take the gradient as $-\frac{1}{3}$.. and .. then you'd
- 10 Trevor: You'd substitute that in, 5 and 2
- 11 Liz: Yep
- 12 Trevor: And then that would give you the perpendicular lines - at the foot - that's at the bottom part - that would be down there (he has drawn a sketch)
- 13 Liz: Do me a whole diagram, Trevor
- 14 Trevor: Which - the perpendicular line (*mumbling*) does it sort of go .. so say that's your line and you wanted to find the coordinates there
- 15 Liz: Right could you put some labels on there for me? Which line is which? ...
- 16 Trevor: And that's whatever that line was that starts with $y =$
- 17 Paul: $3x - 5$
- 18 Liz: And where is A ?
- 19 Trevor: There - no it can't be there. A 's up there
- 20 Paul: I thought A was there

- 21 Trevor: No because you already know the coordinates don't you?
- 22 Liz: Is A on the line $y = 3x - 5$?
- 23 Paul: No, no it's not, yes, so A must be there
- 24 Liz: Okay, do me a reasonably accurate diagram
- 25 Paul: Wouldn't you then, yeah, if you knew the point (*Trevor and Paul both talking together*) you do the intersection if you knew the equation of both lines
- 26 Liz: Yeah
- 27 Paul: you could find where they cut ...
- 28 Liz: I don't think this set-up is entirely clear to all three of us - so let's see if we can get a diagram so that we know what we're all talking about. Can you draw a fairly accurate diagram of what $y = 3x - 5$ looks like? Right, okay And what about the point (5, 2)?
- 29 Trevor: Probably the line would cross through there
- 30 Liz: How can you tell whether that point actually lies on the line or not?
- 31 Paul: Oh no it doesn't work
- 32 Liz: Okay so it's not on the line - it's going to be somewhere below it in fact isn't it? Okay - so that's A. Can you draw in the line that they're first asking you to find the equation of
- 33 Trevor: Where's the question - number 5?
- 34 Liz: Umm. Read it again ..
- 35 Trevor: 'Hence find the coordinates ..'
- 36 Liz: The first line they're asking you to find the equation of - 'find the equation of the line -'
- 37 Trevor: 'through A(5, 2) .. perpendicular to the line '
- 38 Liz: Right okay - so that's the equation - sorry, that's the line you're finding the equation of to start with
- 39 Trevor: right
- 40 Liz: Then, they're then asking you to find the coordinates of the foot of the perpendicular from A to the line - do you know which point that is?

- 41 Paul: No
- 42 Trevor: No
- 43 Liz: Okay, it's the point where the line that you've just found the equation of meets the original line
- 44 Paul: So it's the point of intersection
- 45 Liz: It's the point of intersection .. because this this line that you've just found the equation of is the perpendicular from A to the other line, and the foot of that perpendicular is the point where they cross - so once you've found the equation of the perpendicular line .. in order to do the last part of the question you've got to find where the two lines intersect
- 46 Trevor: Ye-es
- 47 Liz: Do you want to do that question? (*Paul starts working on it*)
- 48 Trevor: I haven't a clue how to find the intersection
- 49 Liz: Well that's what you've just been doing at the end of 9e
- 50 Trevor: Is it?
- 51 Liz: Yeah.
- 52 Trevor: Oh yeah cos you have the two thingies
- 53 Liz: Yeah but you've got to do the first part first, where you find out what the equation of that line is
- 54 Trevor: So if that's 2 that's 5 (*mumbles*) so that'd be (*he starts writing*) minus ..
- 55 Paul: Five over three
- 56 Trevor: Plus a third
- 57 Liz: Um what have you got? It was
- 58 Trevor: Cos you get that it's five that'd have to be -
- 59 Liz: It's $y = ..$ minus a third - is that right - yes - minus a third x plus c
- 60 Trevor: That should be negative five thirds
- 61 Liz: That's got to be two . and that's got to be five
- 62 Trevor: So that gives us two and a third, so that's got to be
- 63 Paul: Three and two thirds, that's what I've got
- 64 Liz: Yeah or eleven over three

65 Trevor: Three and two thirds (*inaudible*)

66 Liz: That's one and two thirds

67 Trevor: Yeah

68 Liz: So two plus one and two thirds .. that's all right .. two and one and two thirds is three and two thirds

69 Trevor: Yeah but if you've already got negative two thirds - yeah - and you plus -

70 Liz: Negative two thirds?

71 Trevor: yeah - negative two into

72 Liz: That's not negative two

73 Trevor: Eh? ..

74 Paul: He's rearranged

75 Trevor: That's a negative five thirds, that is

76 Liz: Yeah, I've swapped it over without telling you really - I've left the c where it is .. and I've left the 2 where it is and moved the five thirds - that's what I've done

77 Trevor: Ye-eh, eleven thirds yeah (*all three continue working*)
.....

78 Paul: It's two point six

79 Trevor: That gives the first one ...

80 Paul: What coordinates did you get for the intercept

81 Liz: Got x is 3 and a quarter

82 Paul; What - from the two equations?

83 Liz: Yeah

84 Paul: I got two point six somehow

85 Liz: Oh! We've done it wrong Trevor - that's minus a third x isn't it?

86 Trevor: Oh yeah - Sugar! - yeah, so that would be a negative x so that would become positive so that would become $10x$

87 Paul: yeah $10x = 26$

88 Trevor: 26 over 10 so yeah, you're right

- 89 Liz: So it's two and three fifths or thirteen over five or two point six or whatever you like - okay, and then we'd have to get y as well
- 90 Trevor: Just put two and three fifths back into one of them (*referring to the equations $y = 3x - 5$ and $y = -\frac{1}{3}x + \frac{11}{3}$*)
- 91 Paul: (*laughs*) The first one's easier!
- 92 Trevor: Yeah!
- 93 Liz: Yeah! (*laughs*) That's thirty nine over five minus -
- 94 Trevor: So that would give you ..
- 95 Liz: Two and four fifths
- 96 Paul: Yeah ..
- 97 Trevor: Yeah
- 98 Liz: Okay, so that's that one - people thought that was really hard the first time we did the chapter on it
- 99 Paul: Once you get what the question's trying to get you to do it's all right - it's just the 'foot of' instead of -
- 100 Liz; Yes - that's true, but in fact when we did this question in the first place - I don't know whether you were there, Paul
- 101 Paul: No, I wasn't
- 102 Liz: I think it was when you were away - um, I did a diagram on the board showing this line and A and the perpendicular and all the rest of it, um, but at that stage your skills with solving equations like this were at the stage where you couldn't guarantee to get to the end of it without making a mistake, and so there are too many other things to go wrong, and it was all a bit too difficult. Now, the fact that you're a lot better at that than you were a few months ago means that the whole thing is much easier to cope with. Any way. Number 6. Oh right.
- (*Number six reads 'Find, in terms of a and b , the coordinates of the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$ '*)
- 103 Trevor: *reads question*
- 104 Liz: I'd start with a diagram, Trevor.
- 105 Trevor: Yeah, I'm baffled straight away, just thinking about it.

Writes $x + 2y = 4$, $2y = -x + 4$, $y = -\frac{1}{2}x + 4$

106 Liz: Plus 2

107 Trevor: Oh yeah, plus 2, because you divide by 2. So that would be the same as like that.

108 Liz: Yeah.

Trevor draws a diagram showing a straight line crossing the y-axis at 2 and having gradient approximately -2

109 Liz: It's not as steep as that because the gradient is only minus a half.

110 Trevor: Oh yeah, it should be more like that.

111 Paul: Why have you got plus 2 there?

112 Trevor: 4 over 2.

113 Paul: No its all right I've got minus 4. Yeah I got half x minus 4 over 2.

114 Liz: It should be plus, I think.

115 Paul: Yeah, yeah it is plus, because it's those two added together.

116 Liz: It's minus a half x plus 2

117 Paul: x can't be zero. x can't be zero can it?

118 Liz: Why not?

119 Paul: Well I suppose it can.

120 Trevor: No, it can't.

121 Paul: If you can have zero times a half.

122 Trevor: But you have got plus two haven't you.

123 Paul: Yeah, x can be zero actually and y is two.

124 Liz: Yeah

125 Trevor: It will never pass through the origin

126 Liz: It doesn't pass through the origin

127 Trevor: *reads the question again* So you have got to find the perpendicular line to that one.

128 Liz: Yeah.

129 Trevor: The point (a, b) which can be anywhere.

130 Liz: Right

- 131 Trevor: If I say that there is (a, b) . *(He places (a, b) on the line $y = -\frac{1}{2}x + 2$)*
I have got to find that point there. *(He draws a perpendicular to the line through (a, b) and indicates a point at the end of it.)*
- 132 Liz: Um, Have a look back at your other diagram. *(I am referring to the last question we worked on)* The point that you were drawing the perpendicular from.
- 133 Trevor: Yeah.
- 134 Liz: It wasn't on the line was it?
- 135 Trevor: It was, about here somewhere, so that would be (a, b) *(indicating a point on the perpendicular some distance from the original line)*. That would be the point that I would be looking for *(pointing to the point of intersection of the original line and the perpendicular through (a, b))*
- 136 Paul: *(to himself)* the gradient is 2
- 137 Trevor: So I want to change that around so the gradient is 2 for that line.
(Writes $y = 2x + c$)
- 138 Liz: Uhummm
- 139 Trevor: I know that the coordinates are (a, b) , so you have got ...
- 140 Paul: b .
- 141 Trevor: Yeah. b equals $2a$ plus c . Yeah?
- 142 Liz: Yeah?
- 143 Paul: Yeah.
- 144 Trevor: There is too many unknowns so you have to leave it like that
(laughs)
- 145 Liz: That's a good point, there are too many unknowns in this question, for you to be able to get an answer with no unknowns in it.
- 146 Paul: That's why they want it in terms of a and b .
- 147 Liz: Yeah, that's right.
- 148 Trevor: So you can... I don't know ... you could work it out without using c I suppose
- 149 Paul: Well, c equals $b - 2a$ doesn't it?

- 150 Liz: Uhuh
- 151 Trevor: Yeah ... You could use that and that I suppose and put that there, but .. *(He indicates putting $c = b - 2a$ into $b = 2a + c$)*
- 152 Liz: Why were you trying to find out c ?
- 153 Trevor: I don't know .. if you find out the point of c it might give you the place where it crosses the y axis but there's no point 'cos you need b and a anyway wouldn't you.
- 154 Paul: Yeah
- 155 Liz: Yeah, that's ..
- 156 Paul: If you know c couldn't you then get rid of c s in the final answer so you've got it all in terms of a s and b s.
- 157 Liz: Hang on a minute, what do you mean by the final answer, Paul?
- 158 Paul: Well when you write out your thing in terms of a and b if you've got rid of c you can have it all in a s and b s
- 159 Liz: Yes
- 160 Paul: So it would be in terms of a and b instead of in terms of a , b and c , 'cos then you've got all your a s and your b s - you've got rid of your c .
- 161 Liz: Right .. you're quite right there but you've just slightly lost track of what it was you were doing here because I think you got confused by the a s and the b s. What you are being asked to do in this question is pretty much the same as you were being asked to do in question five
- 162 Paul: Uhum
- 163 Liz: Except that they haven't given you a real point $(5, 2)$ they have given you the point (a, b) but they're asking you to find the coordinates of the foot of the perpendicular. We did question five in two stages. What was the first stage?
- 164 Paul: Find out the equation of ..
- 165 Trevor: .. the perpendicular line
- 166 Paul: the perpendicular line
- 167 Liz: Right .. and that was why you wanted c - so that you could write down the equation of that perpendicular line - so can you do that?

- 168 Trevor: You could alter this round using that $ax + by + c$ thing. (*Earlier in the session Trevor has mentioned a section in the text book where it is stated that the line $ax + by + c = 0$ is perpendicular to $bx - ay + k = 0$*)
- 169 Liz: Yeah
- 170 Trevor: 'Cos you know what that'll be but you won't know the value of k but the value of k will be (*inaudible*) the opposite of that.
- 171 Liz: Yes try it. k isn't going to be the same thing as c
- 172 Trevor: I don't know
- 173 Liz: But it will be connected to it
- 174 Trevor: You don't have a specific point 'cos if you could put in a point
- 175 Liz: Well you do have a specific point .. 'cos you know the line goes through (a, b)
- 176 Trevor: (a, b) .. yeah so I could put (a, b) in - so it would be - yeah but it would be $2x + y$ and you don't know what k could be - it could be a negative or I don't know .. it's weird
- 177 Liz: So if you want to find out what the value of k is
- 178 Trevor: You need a coordinate so you can put the numbers in so you can prove that that is zero and then you can find from that value of k but if you put a and b in you are just going to come out with
- 179 Liz: something in terms of a and b
- 180 Trevor: Yeah
- 181 Liz: Yeah but that's what your answer's got to be - in terms of a and b
- 182 Trevor: Right - so you'd have $2a + b + k = 0$
- 183 Liz: Uhummm
- 184 Trevor: So $k = -2a - b$
- 185 Liz: Yeah - um, yes I'm afraid I've got to take you back to here because there's a mistake in that, which is why this hasn't come out quite right
- 186 Trevor: Yep
- 187 Liz: You see here you've got plus x and plus $2y$
- 188 Trevor: Yeah
- 189 Liz: So here one of them's got to be negative

- 190 Trevor: That's minus, that one's minus
- 191 Liz: Yeah okay so that makes that minus b
- 193 Trevor: So that'll become plus b
- 194 Liz: Yeah, so in fact your k is the same as the c here
- 195 Trevor: So k is $-2a + b$ which is the same as that there .. so you can now put in the value for k .. $2a - b + b - 2a$.. so you'd have $2a$.. so yeah, that would prove that that equalled zero
- 196 Liz: Yes
- 197 Trevor: If you put that in, because those two would cancel each other out
- 198 Liz: Uhuh
- 199 Trevor: So .. umm .. you would then .. sort of ..
- 200 Liz: What we wanted as an intermediate stage was the equation of the line
- 201 Trevor: Yeah
- 202 Liz: Now you said, .. first we said we're going to have the equation of the line as $y = 2x + c$
- 203 Paul: Yeah
- 204 Liz: and you worked out what c was
- 205 Paul: Yeah
- 206 Liz: So remember the equation of the line is $y = 2x + c$ and now you know what c is
- 207 Paul: So it's $y =$
- 208 Liz: So the equation of the line is $y = 2x + b - 2a$
- 209 Trevor: Yeah - but when you put .. so you'd have $b = 2a + b - 2a$.. weird!
- 210 Liz: Yeah - it's not getting you anywhere - keep putting b and a back in
- 211 Trevor: So I should just leave it as $y =$
- 212 Paul: $2x$.. that's the actual equation
- 213 Trevor: $2x + b - 2a$
- 214 Liz: What you were doing up here with the k s is a different way of doing the same thing
- 215 Trevor: Yeah

- 216 Liz: 'Cos that would have given you the equation $2x - y + b - 2a = 0$ which is the same equation
- 217 Trevor: So now you substitute that into that or that into that (*indicates*
 $y = 2x + b - 2a$ and $y = -\frac{1}{2}x + 2$)
- 218 Liz: Yeah
- 219 Trevor: It's probably better to do that one
- 220 Liz: Well - since they both say $y =$ I would just say so that equals that
- 221 Trevor: Yeah, so you'd have $2x + b - 2a = -\frac{1}{2}x + 2$
- 222 Liz: Yeah
- 223 Trevor: Three halves of x
- 224 Liz: Er, five halves isn't it? Does that say three and a half?
- 225 Trevor: Yeah - oh, no it's two and a half - two and a half x plus b minus $2a$ equals 2 and that's it
- 226 Liz: Well, more or less, I would double that all the way through so I get rid of
- 227 Trevor: the half
- 228 Liz: half
- 229 Trevor: Two x
- 230 Liz: Five x if you double it
- 231 Trevor: Yeah five x sorry plus $2b$ minus $4a$ equals 4
- 232 Liz: Okay - and you've got to solve that for x - so you've got to get x equals
- 233 Trevor: So $4a - 2b + 4$.. so x would equal all of that over 5
- 234 Paul: Where did you get the plus 4 at the end from
- 235 Trevor: That's just where it's always been innit? You got the 2 there
- 236 Paul: Oh yeah
- 237 Trevor: Then when you times it
- 238 Paul: I see yeah
- 239 Liz: Okay so if you now go back to one of the equations and work out y by putting that in for x , it's pretty unpleasant, it's a bit messy, but if you do that

240 Paul: You'll come out, yeah
241 Liz: You then get a value for x
242 Paul: and a value ..
243 Liz: and a value for y in terms of a and b which is what they asked
you for.

Appendix E Invitation Letter

*Centre for Mathematics Education
Open University
Milton Keynes
MK7 6AA
Tel: 0908 – 652136*

September 16th 1994

Dear Colleague,

I am writing to tell you about an opportunity which you or other members of your department may wish to take up to join a study group on 'A' level pure maths teaching.

I was head of maths at a comprehensive school in Berkshire until the end of 1992 and am now a full-time research student at the Open University. I am researching into sixth form students' understanding of variables. I hope that what I have to say about my research would be of interest and of use to you in your teaching. During last year I shared the teaching of a lower sixth pure maths class at (school A) with their usual teacher and I have used my detailed observations of his class, together with ten years' experience of teaching maths, to put together some material concerning the role of generality in 'A' level.

What I am asking of you is to meet with me after school for roughly an hour and a half on five or six occasions over the next term or so. During this time I would like to tell you about some of the incidents which I have recorded from my teaching and to involve you in some activities which I hope will enable us to understand better the kind of difficulties that students have with notions of variation and generality. We could also discuss some of the teaching gambits which I have used to try to focus students' attention on these ideas.

The purpose of our meetings for me would be to try out some ways of putting across to other people what I have gained from last year and to find out your reactions to what I have to say. I will want to know whether the ideas ring true with your own experience in the classroom and whether the activities I suggest to you help you to get a sense of what it is like for the students. If it were

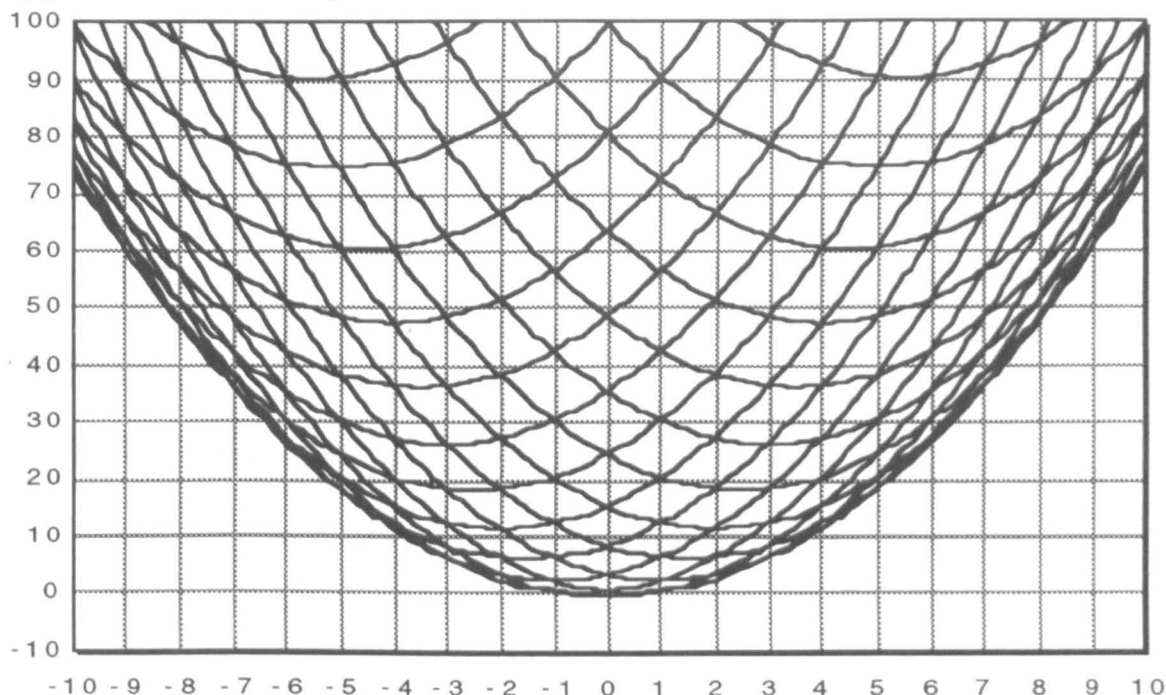
possible I would also like to visit your 'A' level class some time during the term, either to teach or observe or both as you wish.

Two of the staff at (school A) have already expressed an interest in joining me for these sessions. They are both teaching the pure maths component of 'A' level for the first time this year. It's likely that the meetings would be most useful to people in the same sort of position but more experienced teachers would also be very welcome. We have tentatively arranged a first meeting for Wednesday 28 September after school at (school A). There is no obligation to meet again after this first time if you don't think it will be useful for you!

If you or one or more of your colleagues is interested in joining the group you can write to me at the OU address above or phone me on that OU number during the day. You can also reach me on 0734 471594 any evening except Wednesday and at weekends.

Thanks for taking the time to read this, and I look forward to hearing from you.

Appendix F Envelopes



GraphPlot II ©1993 by Graham Cox

Fig 1

This diagram shows a set of curves which are graphs of the equation $y = x^2 + ax + a^2$ for integer values of a from -11 to 11 . Taken together these parabolae appear to generate another parabola. I will call this apparent parabola the envelope of this set of curves.

How can I find the equation of that parabola? Would any other families of parabolae generate the same envelope? Could I have predicted that this set of curves would have an envelope, and if so that it would be a parabola? What kind of envelopes might be generated by families of other types of curves?

First assume that the envelope is a curve which is tangent to each of the set of curves. That is that it has the same y -value and the same gradient at a point as each member of the generating set.

I will try to find the equation of the envelope in my diagram by assuming that it is a parabola. Because each of the set of generating curves is a parabola, the coordinates of the point at which each meets the envelope will be the solution of a quadratic equation. I can use my special knowledge of quadratic equations to say that two quadratic curves are tangent to each other if solving their equations simultaneously yields an equation with a repeated root. I am assuming only that the envelope is a parabola, so I will give it equation $y = px^2 + qx + r$. So I want to find values of p , q and r for which $y = px^2 + qx + r$

and $y = x^2 + ax + a^2$ have a repeated root when solved simultaneously for every value of a .

Put $px^2 + qx + r = x^2 + ax + a^2$

Then $x^2(p-1) + x(q-a) + (r-a^2) = 0$

This equation must have equal roots for every value of a so

$$(q-a)^2 - 4(p-1)(r-a^2) = 0$$

$$q^2 - 2aq + a^2 - 4(pr - pa^2 - r + a^2) = 0$$

$$a^2(1 + 4p - 4) - 2aq + (q^2 - 4pr + 4r) = 0$$

Since this equation must hold for all values of a , the coefficients of a^2 , of a and the constant term must each be zero:

$$1 + 4p - 4 = 0 \quad \Rightarrow \quad p = \frac{3}{4}$$

$$2aq = 0 \quad \Rightarrow \quad q = 0$$

$$q^2 - 4pr + 4r = 0 \quad \Rightarrow \quad r = 0$$

So the equation of the envelope is $y = \frac{3}{4}x^2$. This is confirmed by checking from the diagram that the envelope has its turning point at the origin and passes through (10, 75).

Is it possible to work back the other way i.e. to find the equations of the set of curves that would generate the parabola $y = \frac{3}{4}x^2$ as an envelope? Suppose I specify that I am looking for a set of parabolae of which $y = \frac{3}{4}x^2$ is the envelope. I need also to decide how I will represent that this is a family of curves. I think their coefficients must be a function of some parameter. Perhaps there could be more than one parameter. I will assume not to start with. So a member of the family that I am looking for will be:

$$y = f(a)x^2 + g(a)x + h(a)$$

I'm concerned that I will not be able to define so many arbitrary functions. I have given a form to this curve but I seem to have no information about it.

If this curve touches the curve $y = \frac{3}{4}x^2$ for each value of a then:

$$f(a)x^2 + g(a)x + h(a) = \frac{3}{4}x^2$$

has repeated roots for all values of a .

$$x^2(f(a) - \frac{3}{4}) + g(a)x + h(a) = 0$$

has repeated roots if

$$(g(a))^2 - 4h(a)\left[f(a) - \frac{3}{4}\right] = 0$$

Since this must be true for all a , is it true that each term must be zero i.e. that $g(a)$ must be zero and that either $h(a)$ or $\left[f(a) - \frac{3}{4}\right]$ must be zero? No, because I am dealing with functions, not constants. To say that $g(a)$ must be zero is to say that it must be the zero function. In fact I cannot simplify the statement any more than I have done. If I decide on functions for $f(a)$ and $h(a)$, then $g(a)$ is determined.

The equations with which I started have $f(a) = 1$, $g(a) = a$ and $h(a) = a^2$, so they fit the condition which I have arrived at. Let me try out some more:

Suppose $f(a) = 0$ (so that the family is of straight lines, not curves) and $h(a) = -3a^2$ (so that $g(a)$ is easily calculated), then $g(a)$ must be $3a$.

So some lines in this family are:

$$a = -4 \quad y = -12x - 48$$

$$a = -3 \quad y = -9x - 27$$

$$a = -2 \quad y = -6x - 12$$

$$a = -1 \quad y = -3x - 3$$

$$a = 0 \quad y = 0$$

$$a = 1 \quad y = 3x - 3$$

$$a = 2 \quad y = 6x - 12$$

$$a = 3 \quad y = 9x - 27$$

$$a = 4 \quad y = 12x - 48$$

This set of straight lines also appears to produce the parabola $y = \frac{3}{4}x^2$. (See fig 2 overleaf)

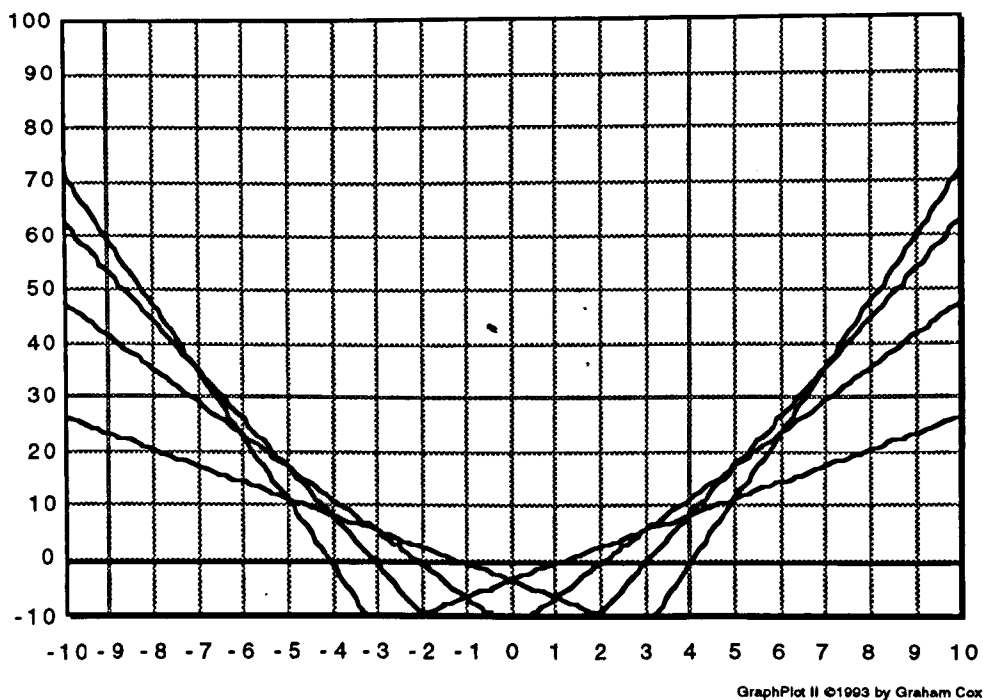


Fig 2

Suppose now that $f(a) = a$ and $h(a) = a$, then $(g(a))^2 = 4a(a - \frac{3}{4})$ and the family of curves is given by

$$y = ax^2 + 2x\sqrt{a(a - \frac{3}{4})} + a$$

There is no member of this family for $0 < a < \frac{3}{4}$. Try some values of a from outside of this range:

$a = 0$	$y = 0$
$a = 1$	$y = x^2 + x + 1$
$a = 2$	$y = 2x^2 + x\sqrt{10} + 2$
$a = 3$	$y = 3x^2 + x\sqrt{27} + 3$
$a = 4$	$y = 4x^2 + x\sqrt{52} + 4$
$a = -1$	$y = -x^2 + x\sqrt{7} - 1$
$a = -2$	$y = -2x^2 + x\sqrt{22} - 2$
$a = -3$	$y = -3x^2 + x\sqrt{45} - 3$
$a = -4$	$y = -4x^2 + x\sqrt{76} - 4$

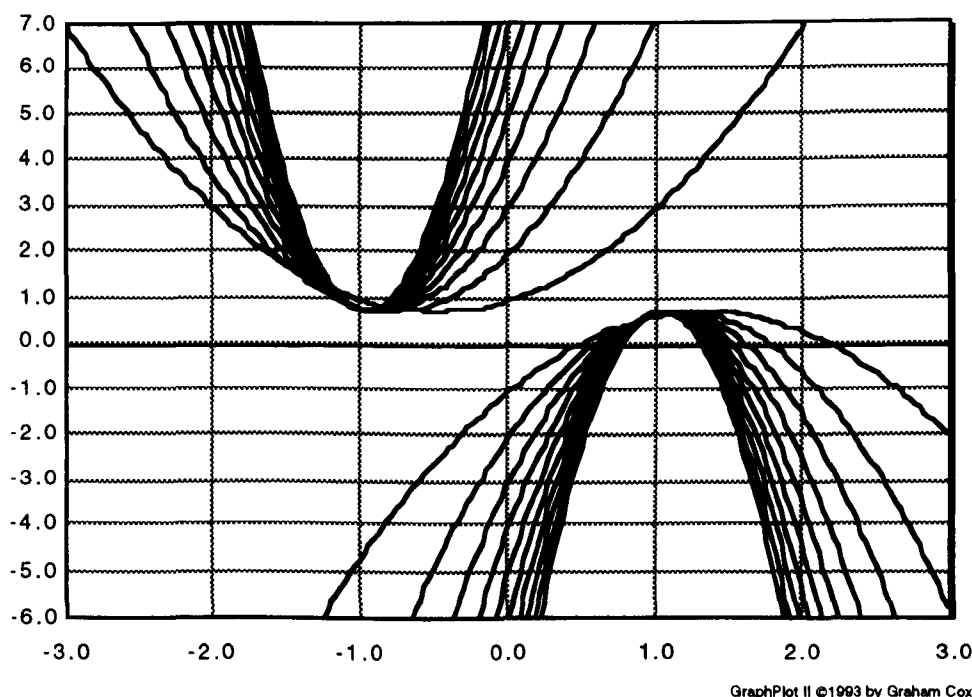


Fig 3

A look at the graphs of these functions confirms that they do all touch the curve $y = \frac{3}{4}x^2$ but they certainly do not make it 'appear'. What additional condition is necessary for the family of curves to make the envelope appear? It seems to be something to do with two things - that the points of contact between the envelope and each member of the family be sufficiently apart and that the members of the family be all on the 'same side' of the envelope. I'll have a look at the condition on the points of contact first.

Where are the points of contact for this last family

$$y = ax^2 + 2x\sqrt{a(a - \frac{3}{4})} + a?$$

The x coordinates satisfy

$$ax^2 + 2x\sqrt{a(a - \frac{3}{4})} + a = \frac{3}{4}x^2$$

$$\text{i.e. } x^2(a - \frac{3}{4}) + 2x\sqrt{a(a - \frac{3}{4})} + a = 0$$

Since this equation has repeated roots, the solution must be

$$\frac{-2\sqrt{a(a - \frac{3}{4})}}{2(a - \frac{3}{4})} = \sqrt{\frac{a}{a - \frac{3}{4}}} = \sqrt{1 + \frac{3}{4a - 3}}$$

But this will not be very far from 1 unless $4a - 3$ is close to zero, i.e. a is close to $\frac{3}{4}$. I know that a cannot be between 0 and $\frac{3}{4}$ so I will try values just above $\frac{3}{4}$.

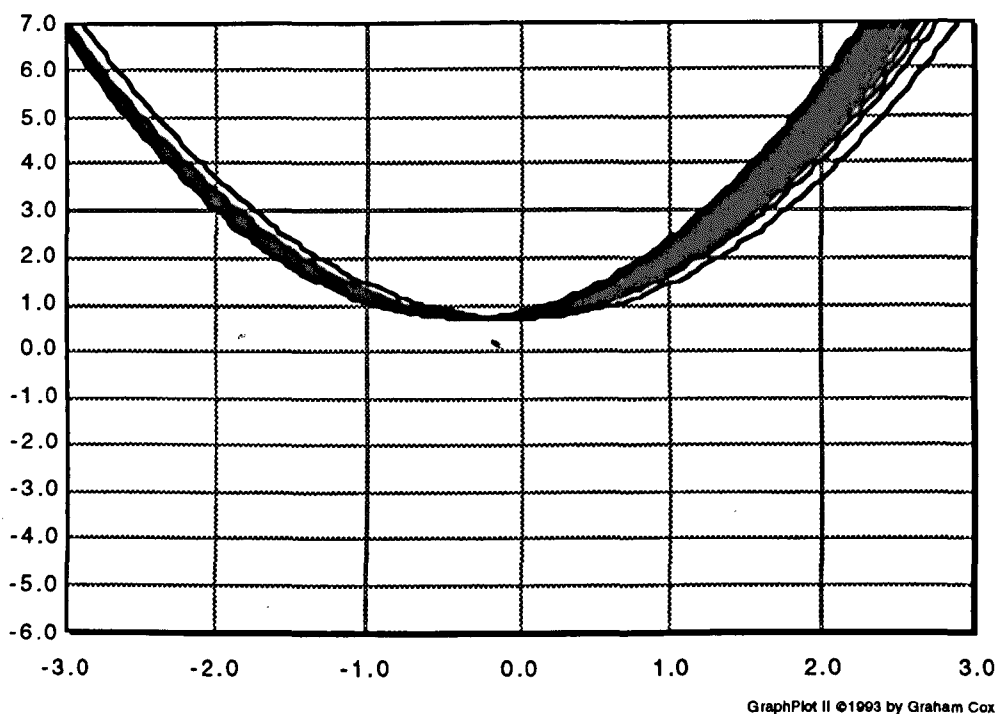


Fig 4

Close to $a = \frac{3}{4}$, the point of contact moves a lot with a small change in the value of a , but the curve itself does not change very much, so that the curve $y = \frac{3}{4}x^2$ still does not 'appear'.

Perhaps a better picture appears if the x -coordinate of the point of contact is a linear or quadratic function of a rather than a reciprocal function. I will explore this idea.

For the general family of functions

$$y = f(a)x^2 + g(a)x + h(a)$$

the x coordinate of the point of contact is the repeated solution to the equation

$$x^2(f(a) - \frac{3}{4}) + g(a)x + h(a) = 0$$

i.e. it is

$$\frac{-g(a)}{2(f(a) - \frac{3}{4})}$$

This will be linear if the order of $g(a)$ is one more than the order of $f(a)$. But

$$(g(a))^2 - 4h(a)[f(a) - \frac{3}{4}] = 0$$

so I can think of several possibilities:

$f(a)$ is constant, $h(a)$ quadratic and $g(a)$ linear

$f(a)$ is linear, $h(a)$ cubic and $g(a)$ quadratic

$f(a)$ is quadratic, $h(a)$ quartic and $g(a)$ cubic

etc.

To begin try $f(a)$ constant. $f(a) = \frac{3}{4}$ would be straightforward but unproductive.

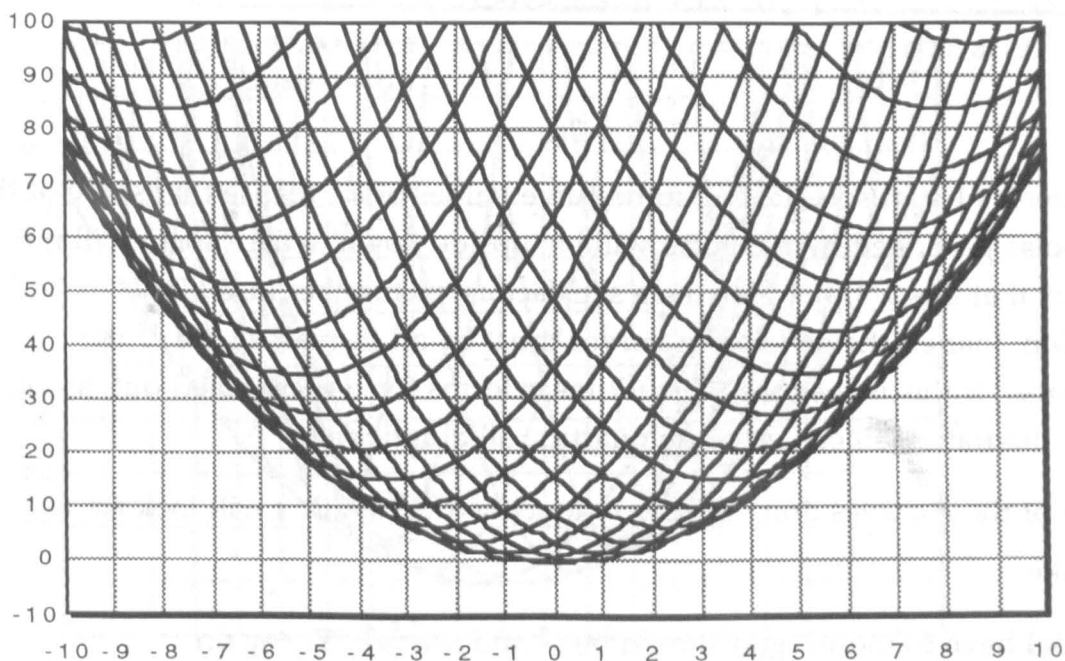
$f(a) = 1$ would give me the set of curves I already have, no matter what quadratic function I chose for $h(a)$. Suppose $f(a) = \frac{7}{4}$. Let $h(a)$ be a^2 for simplicity and without loss of generality

(I've just realised that this notion of getting the same set of curves regardless of choice of $h(a)$, having once chosen $f(a)$ is important. It reduces the sense of arbitrariness of sets of curves, because it means that it is the representation of the set in parametric form which is responsible for some of the arbitrariness. So for instance $y = x^2 + ax + a^2$ represents the same family of curves as $y = x^2 + (a + 1)x + (a^2 + 2a + 1)$. What kind of awareness helps me to see that my choice of $h(a)$ doesn't matter?)

Then, with $f(a) = \frac{7}{4}$ and $h(a) = a^2$, $g(a) = 2a$ (I should include $-2a$ as a second possibility but that would only generate the same set of curves). So I have a new set of curves to try:

$$y = \frac{7}{4}x^2 + 2ax + a^2$$

This gives a very satisfactory result.



GraphPlot II ©1993 by Graham Cox

Fig 5

I am confident that I can generate a 'nice-looking' set of curves for every constant value of $f(a)$ I might choose.

Now what about linear functions for $f(a)$?

To make my working easy I will start with $f(a) = a + \frac{3}{4}$. Then $h(a)$ must be a cubic, $\frac{1}{4}a^3$ say. Finally $g(a)$ must be $\frac{1}{2}a^2$. So my family of curves is

$$y = (a + \frac{3}{4})x^2 + \frac{1}{2}a^2x + \frac{1}{4}a^3$$

Let me try that.

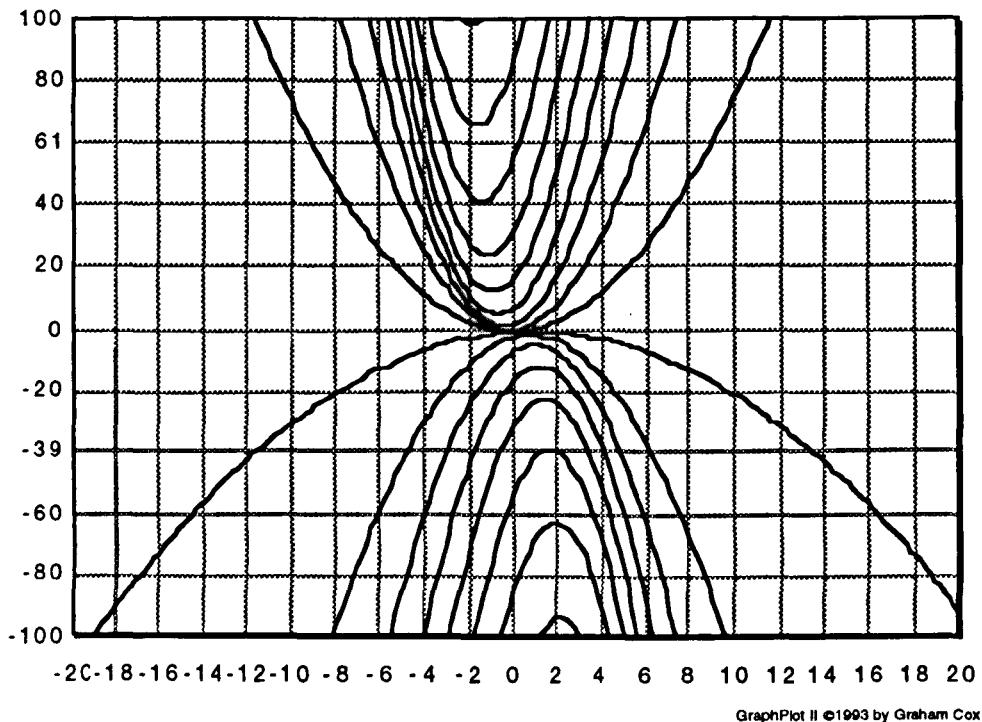


Fig 6

(As soon as I start substituting in negative values of a I realise that some of the parabolas generated by this form will be 'upside down' where some will not. I suspect that this will not give me a satisfactory picture because of my second criterion, that all the generating curves must be on the same side of the envelope. If this is the case then all linear forms for $f(a)$ are ruled out, as are some quadratics, all cubics, some quartics, all quintics etc.)

Many of these curves don't appear to touch $y = \frac{3}{4}x^2$ at all. I will look for a mistake.

I think I have found it! $g(a)$ should be a^2 and not $\frac{1}{2}a^2$. Try again.

(When I first wrote the above sentence it read ' $g(a)$ should be x^2 and not $\frac{1}{2}x^2$.' I re-read it several times before noticing the mistake.)

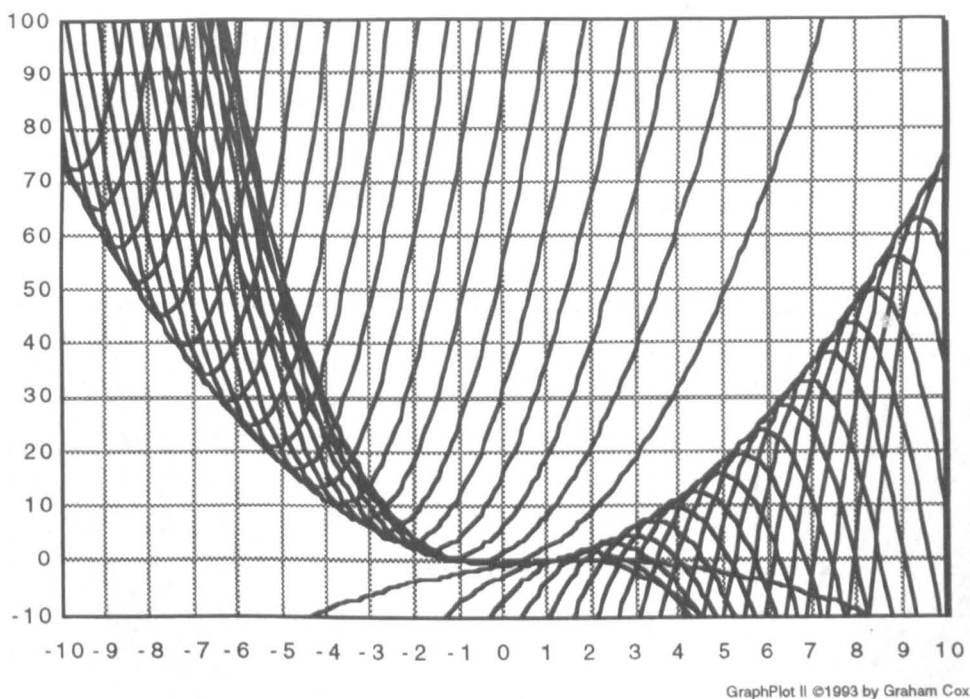


Fig 7

These curves are all tangent to $y = \frac{3}{4}x^2$ but they map it out in rather a different way because they are on both sides of the curve.

What about a quadratic function for $f(a)$? The easy one would be $a^2 + \frac{3}{4}$. Then if $h(a)$ is a^4 and $g(a)$ is $2a^3$. So I will try

$$y = (a^2 + \frac{3}{4})x^2 + 2a^3x + a^4$$

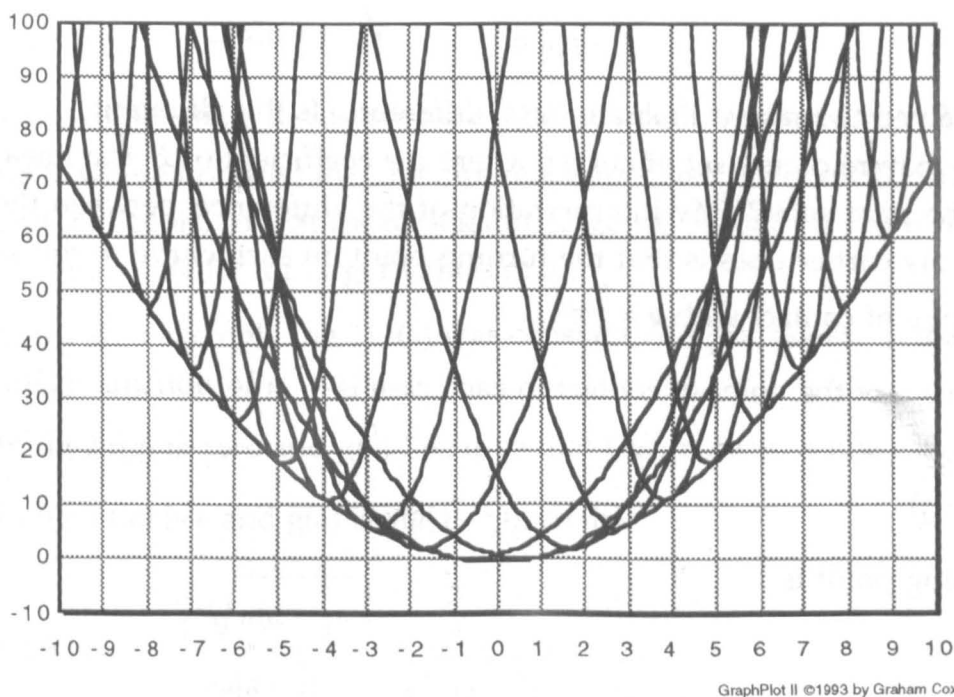


Fig 8

This seems quite different. The curves are steeper and they all touch $y = \frac{3}{4}x^2$ close to their turning points. Also they appear not to overlap each other so much I think they would look less 'different' if I chose different values of a for the curves I plot. This picture used integer values of a from -10 to 10. I will try values of a between -1 and 1.

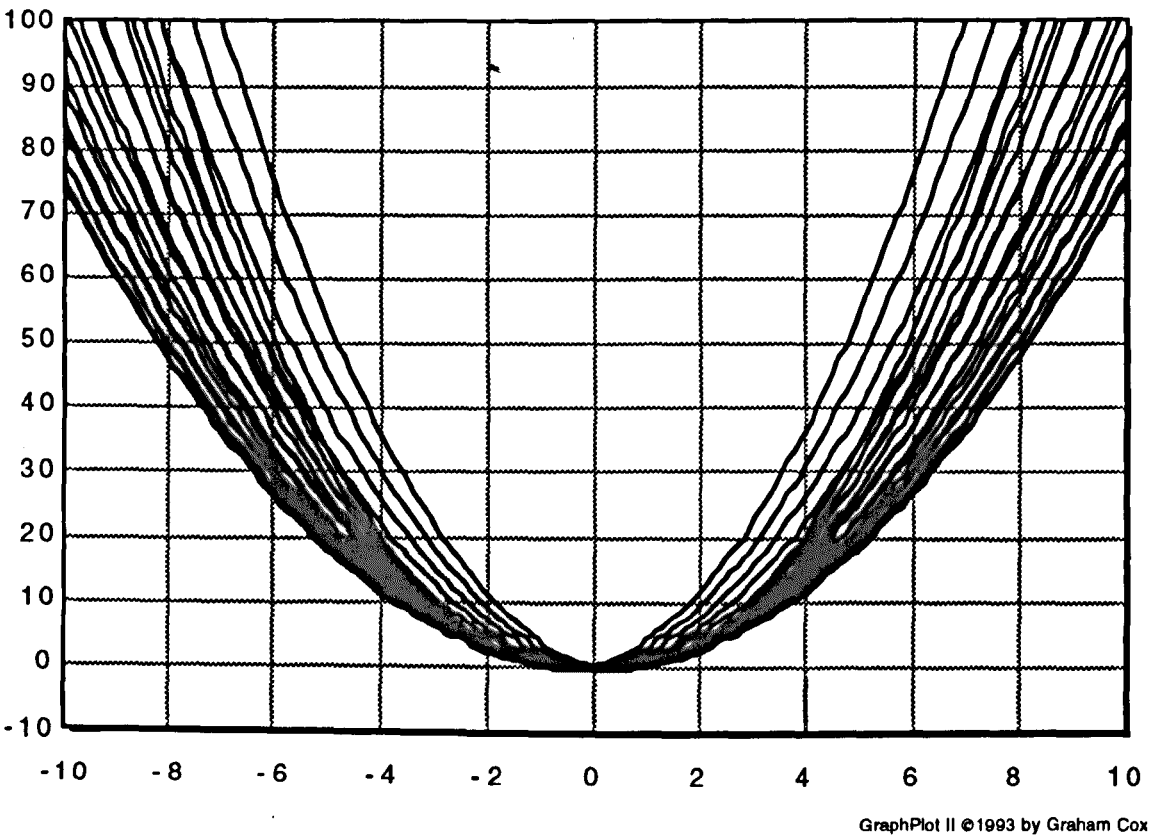


Fig 9

This hasn't helped to make it look any less 'different'. Is this different appearance a feature of any set of curves where the coefficient of x^2 is a linear function of the parameter? My interpretation of the 'difference' between these pictures and my earlier ones is that the turning point on each curve in the set is close to its point of contact with $y = \frac{3}{4}x^2$.

The x -coordinate of the point of contact in each case is

$$\frac{-g(a)}{2(f(a) - \frac{3}{4})}$$

and the turning point is

$$\frac{-g(a)}{2f(a)}$$

The difference between these two values will be small provided $\frac{3}{4}$ is small compared to $f(a)$. This will be true for a greater range of values of a when $f(a)$ is a higher order polynomial. If $f(a)$ is constant then the difference is constant - no it's not. I am again thinking of $g(a)$ as constant within the set of curves whereas in fact it varies with a . I'll try to clear that up by looking at an expression for the difference.

This reasoning also predicts that the turning point will be close to the point of contact for those members of a set of curves for which $|f(a)|$ is large. I'll look back to an earlier picture to see whether this is true in a case where $f(a)$ is linear.

I keep finding myself confused between the actual magnitude of the two x -coordinates and the difference between them. Perhaps it would help me to work out an expression for the difference so that I have something more 'tangible' to focus on.

The difference between the x -coordinates of the point of contact and the turning point for any curve in the set is

$$\begin{aligned} D(a) &= \frac{-g(a)}{2f(a)} - \frac{-g(a)}{2(f(a) - \frac{3}{4})} \\ &= \frac{-g(a)[f(a) - \frac{3}{4} - f(a)]}{2f(a)[f(a) - \frac{3}{4}]} \\ &= \frac{\frac{3}{4}g(a)}{2f(a)[f(a) - \frac{3}{4}]} \end{aligned}$$

This will be small in cases where $\frac{3}{4}g(a)$ is small compared with $2f(a)[f(a) - \frac{3}{4}]$. In fig 8, where I noticed that the difference appeared to be small, $f(a)$ was quadratic and $g(a)$ cubic, so that for reasonably large values of a this condition holds. Also $D(a)$ gets smaller for larger values of a .

In my first example (fig 1) $f(a)$ was constant and $g(a)$ linear, so that $D(a)$ was a linear function of a .

In fig 2 $f(a)$ is zero and $g(a)$ linear so that $D(a)$ is again a linear function.

In fig 3 $f(a)$ is a and $g(a)$ is $\sqrt{a(a - \frac{3}{4})}$, so that

$$D(a) \text{ is } \frac{\frac{3}{4}\sqrt{a(a - \frac{3}{4})}}{2a[a - \frac{3}{4}]} = \frac{\frac{3}{4}}{2\sqrt{a[a - \frac{3}{4}]}} = \frac{3}{8\sqrt{a[a - \frac{3}{4}]}}$$

Thus the turning point and point of contact are close together for almost all values of a . This is confirmed by the picture.

In fig 5 $D(a)$ is again linear since $f(a)$ is constant and $g(a)$ linear. In fig 7 $f(a)$ is linear and $g(a)$ quadratic, so that for large a $D(a)$ is approximately constant. This looks feasible.

I also find confusion in remembering that the magnitude of the difference will vary with a within a set of curves as well as between sets.

What happens if the x -coordinates of the points of contact are a quadratic function of a ? I could achieve this by having

- $f(a)$ constant
- $g(a)$ quadratic
- $h(a)$ therefore cubic

Let me try $f(a) = 1, h(a) = a^4$ and $g(a) = a^2$. But this will give me the same set of curves as

$$y = x^2 + ax + a^2$$

as would any other set with $f(a) = 1$.

Is it true that any set of curves is essentially determined by the choice of $f(a)$? No I still have real choice over $g(a)$ and $h(a)$ with respect to their relative orders. Or do I? Previously I determined that $g(a)$ should be linear and $h(a)$ quadratic, now I am choosing $g(a)$ quadratic and $h(a)$ quartic and I am achieving essentially the same set of curves.

Maybe my choice of $f(a)$ only determines the set of curves if $f(a)$ is constant. I will repeat my linear choice for $f(a)$ to see if I get a distinct set of curves with my new criterion on the order of the x -coordinates of the point of contact.

So I choose $f(a) = a + \frac{3}{4}, h(a) = a^5$ and $g(a) = 2a^3$. Previously my choice was $f(a) = a + \frac{3}{4}, h(a) = \frac{1}{4}a^3$ and $g(a) = a^2$. For this choice $D(a)$ is

$$\frac{\frac{3}{4}a^2}{2a + \frac{3}{4}[a + \frac{3}{4} - \frac{3}{4}]} = \frac{\frac{3}{4}a^2}{2a + \frac{3}{4}a} = \frac{3a^2}{11a} = \frac{3}{11}a$$

A linear function for $D(a)$ has given good results up till now!

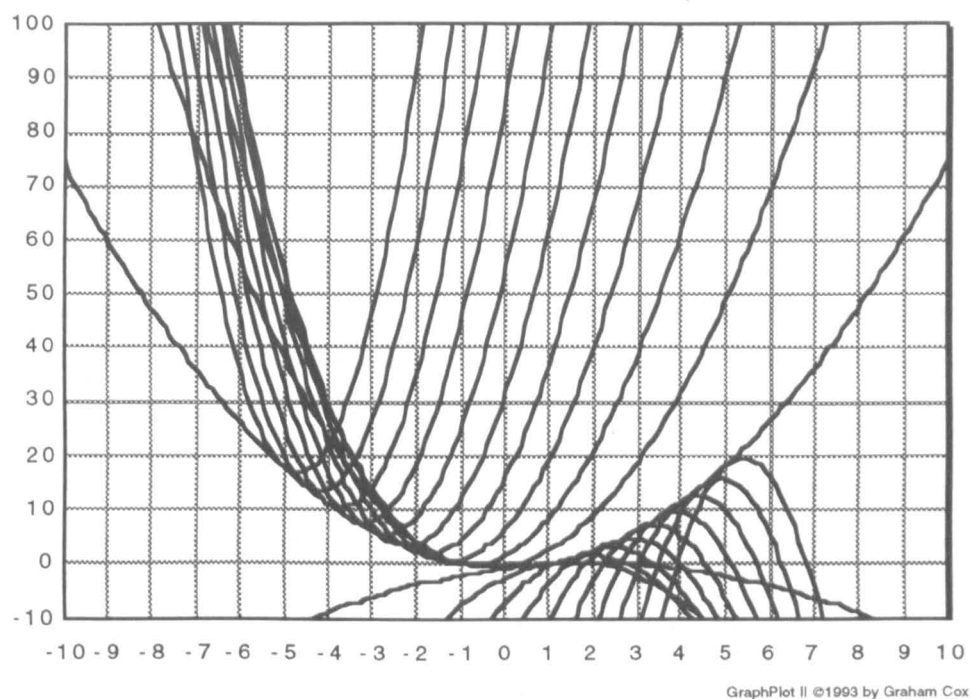


Fig 10

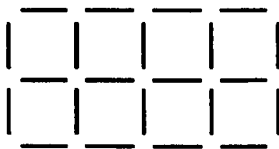
This looks very like fig 7. Should I suspect that I have the same family of curves here?

Appendix G Problem List and Tasks Chart

Problem A For which values of k is $k(k - 1)x^2 + 2(k + 3)x + 2$ positive for all real values of x ?

Problem B Suppose that n (≥ 4) points are chosen on a circle, and each pair of points is joined by a straight line. Assume that no three lines meet at a point except on the circle. Let p_n be the total number of triangles formed within the circle. Find a formula for p_n for $n > 6$

Problem C The picture below shows a rectangle made up of two rows of four columns and of squares outlined by matches. How many matches would be needed to make a rectangle with R rows and C columns?



Problem D Find the equation of a straight line which has gradient M and passes through the point (p, q)

Problem E Factorise

- (1) $x^3 - 1$ (2) $x^3 - 8$ (3) $x^3 - a^3$

Problem F In how many ways can n 1 by 2 rectangles be arranged to form a 2 by n rectangle?

Problem G Prove that affine transformations map straight lines to straight lines.

Problem H A point P , coordinates (a, b) , is equidistant from the x -axis and the point $(3, 2)$. Find a relationship connecting a and b .

Problem I Find the Cartesian equation of the set of points P , if P is equidistant from the point $(4, 1)$ and the line $x = -2$.

Problem J Find, in terms of a and b , the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$.

Problem K Explore sequences generated by the rule $x_{n+1} = ax_n + 1$ for various values of a .

Problem L In the following equation x is an unknown and m is a parameter: $m(x - 5) = m + 2x$. For what value of the parameter m will the equation have no solution?

Problem M Sketch $y = x(x - a)$

Problem N Find the equation of a straight line which passes through the point (m, c) .

Problem O The normal to the curve $y^2 = 4ax$ at the point $P(ap^2, 2ap)$ meets the curve again at Q . Find the locus of the mid-point of PQ .

Problem P The point (a, b) is equidistant from the x -axis and the point $(1, 2)$. Find an equation linking a and b .

Problem Q A circle has centre $(2, 4)$ and passes through the point $(-1, 5)$. The point (p, q) lies on the tangent which touches the circle at $(-1, 5)$. Find an equation linking p and q . Hence write down the equation of the tangent.

Problem R Find the equation of the tangent to the curve $y = x^2 + 1$ which passes through the origin.

Problem S Find the points on the curve $y = e^x$ at which the tangent goes through the origin. Find the points on the curve $y = e^{2x}$ at which the tangent goes through the origin. Repeat for $y = e^{3x}$, $y = e^{3.5x}$, and generalise. The set of points you found appear to form a 'curve'. What curve is it, and why?

Problem	Worked on by	In chapter
A	a class I taught full time in 1992	one
	two classes taught by a colleague	one
	a pair of students from my colleague's class	five
	an individual student from my colleague's class	seven
B	myself	five
C	myself	five
	myself	six
D	Frank	five
	Frank and Paul	six
	Frank	seven
	Eddie	seven
	Tommy	seven
	Paul	seven
E	Trevor and Frank	five
	Frank	five
	Tommy	six
F	teachers' group	five
G	teachers' group	five
	myself	seven
H	Kevin and Trevor	five
I	Trevor	five
J	Eddie	five
	Eddie	six

J	Lorne	six
	Robert	six
	Trevor	six
	Trevor and Paul	seven
	Eddie	seven
K	colleagues, teachers and students	six
L	myself	six
M	Eddie	six
	Lorne	six
	Sam	seven
N	colleagues	six
O	myself	six
P	Lorne	six
Q	class at school A	six
R	Robert	six
	Paul	eight
	Neil	eight
	Sam	eight
	Tommy	eight
S	students at sixth form day	eight
	teachers' groups	eight

Appendix H Outline Solutions

- 1 **Question** A(1, 3), B(5, 7), C(4, 8) and D(a, b) form a rectangle ABCD. Find a and b.

Solution Find the vector from C to B $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$ and add to the coordinates of A to find coordinates of D.

- 2 **Question** The triangle ABC has its vertices at the points A(1, 5), B(4, -1) and C(-2, -4)

(a) Show that ΔABC is right-angled

(b) Find the area of ΔABC

Solution (a) Find lengths AB, BC and AC. Use Pythagoras' rule (the two shortest lengths squared, added together and square rooted). If this gives you the length you have not used yet, the triangle is right angled.
(b) Take the two shortest lengths, times them together and divide by two.

- 3 **Question** Show that the point $(-\frac{32}{3}, 0)$ is on the altitude through A of the triangle whose vertices are A(1, 5), B(1, -2) and C(-2, 5).

Solution Show that the line joining A to the point $(-\frac{32}{3}, 0)$ is perpendicular to the line joining B to C.

- 4 **Question** Show that the triangle whose vertices are (1, 1), (3, 2), (2, -1) is isosceles.

Solution Call the three points (A_1, A_2) , (B_1, B_2) , (C_1, C_2) .

$$B_2 - A_2 = a$$

$$B_1 - A_1 = b \quad a^2 + b^2 = h_1^2 \quad \sqrt{h_1^2} = h_1$$

$$C_1 - A_1 = c$$

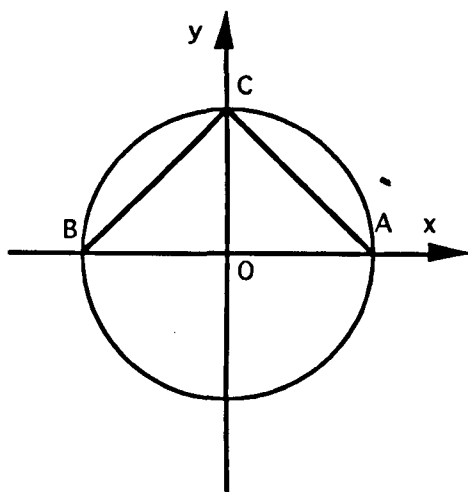
$$C_2 - A_2 = d \quad c^2 + d^2 = h_2^2 \quad \sqrt{h_2^2} = h_2$$

If $h_1 = h_2$, triangle is isosceles.

If not, calculate h_3 . If $h_3 = h_1$ or h_2 triangle is isosceles.

- 7 **Question** A circle, radius 2 and centre the origin, cuts the x -axis at A and B and cuts the positive y -axis at C. Prove that $\angle ACB = 90^\circ$.

Solution



Find the coordinates of A, B and C.

Find the lengths of AC and BC. Use Pythagoras' rule to see if $AB^2 = AC^2 + BC^2$. If it is, then $\angle ACB$ is 90° .

- 10 **Question** ABCD is a quadrilateral where A, B, C and D are the points $(3, -1)$, $(6, 0)$, $(7, 3)$ and $(4, 2)$. Prove that the diagonals bisect each other at right angles and hence find the area of ABCD.

Solution Multiply together the gradients of the lines AC and BD. If they multiply to -1 then the lines must cross at right angles. Find the mid-point of AC and of BD. If they are both the same point then the lines intersect at the centre of each line i.e. they bisect each other.

Calculate the length of the line joining A to C. Call it line 1. Calculate the length of the line joining B to D. Call it line 2. Multiply the length of line 1 by the length of line 2 and divide the answer by 2. This is the area.

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